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Exercises Singularity Theory

1. *(3 points)* Let k be a field, and consider a commutative k-algebra R together with a (descending) filtration $R = I_0 \supset I_1 \supset I_3 \supset \dots$ by ideals (also written as $(I_\bullet) \subset R$, or $(I_\bullet R)$ where the symbol $\bullet$ stands for any index $k \in \mathbb{N}$). Let J be an ideal such that the quotient R/J is a finite-dimensional k-vector space. Prove that

$$\dim_k(R/J) = \sum_{i \geq 0} \dim_k \left(\frac{I_i + J}{I_{i+1} + J} \right).$$

   Hints:

   (a) Show that for all i, there is an isomorphism of R-modules

$$\frac{I_i}{I_i \cap J} \cong \frac{I_i + J}{J}$$

   (b) Show that by considering the quotient map $R \twoheadrightarrow R/J$, the filtration $I_\bullet$ induces a filtration $(\widetilde{I}_\bullet R/J)$ on R/J by k-sub-vector spaces.

   (c) Show that for finite-dimensional k-vector space V and any filtration $F_\bullet V$ by sub-vector spaces, we have $\dim_k(V) = \dim_k \mathrm{gr}^F_\bullet V$, where

$$\mathrm{gr}^F_i V := F_i V / F_{i+1} V \qquad \text{and} \qquad \mathrm{gr}^F_\bullet V = \oplus_{i \geq 0} \mathrm{gr}^F_i V.$$

2. *(3 points)* Do the following germs have isolated singularities ? If yes, determine their Milnor number and the smallest number k such that f is k-determined (i.e., their determinacy).

   (a) $f = x^4 + y^3 \in \mathcal{R}_2$

   (b) $f = x^3 + y^5 + x \in \mathcal{R}_2$.

   (c) $f = x^2 - y^2 \in \mathcal{R}_3$

3. *(3 points)* Compute the Milnor number of the following germs.

   (a) $f = x^3 + xy^p \in \mathcal{R}_2$, $p > 2$,

   (b) $f = x^p + y^p + x^2y^2 \in \mathcal{R}_2$, $p > 3$,

   (c) $f = x^2 + y^2 + 2xy \in \mathcal{R}_2$.

To be handed in until Thursday, 28th May 2020, (by email to `valeria.bertini@mathematik.tu-chemnitz.de`**)**