Exercises Singularity Theory

1. *(2 points)* Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2$.
   (a) Show that f has a non-isolated singularity at 0.
   (b) Show that f is not k-determined, for every $k \geq 0$.
   Hint: For any $l \in \mathbb{N}$, consider the function $g_l : \mathbb{R}^2 \to \mathbb{R}$, $g_l(x, y) = x^2 - y^{2l}$.
2. *(4 points)* Denote by $\mathcal{G}$ the set of germs of local diffeomorphisms/local biholomorphisms ϕ at the origin in $\mathbb{R}^n/\mathbb{C}^n$ leaving the origin invariant.
   (a) Check that $\mathcal{G}$ is a group under the composition, and that
   $$\begin{aligned} \mathcal{R} \times \mathcal{G} &\to \mathcal{R} \\ (f, \phi) &\mapsto f \circ \phi \end{aligned}$$
   is a well-defined right group action on $\mathcal{R}$ (in the differentiable and holomorphic case).
   (b) Note that $f \sim g$, as defined in Lecture 6, if and only if f and g are in the same orbit under the action of $\mathcal{G}$; in this case, we say that f is right-equivalent to g. On the other hand, we say that f and g in $\mathcal{R}$ are k-equivalent if their Taylor polynomials T_f^k and T_g^k are the same. What can we say about relation between equivalence and k-equivalence?
   Hint: Look at the germ of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2(e^x - 1)$, and compare it with the germ of the identity function on $\mathbb{R}$.
   (c) Show that the ideal $\mathfrak{m}^\infty$ in $\mathcal{R}$ is preserved by $\mathcal{G}$, and conclude that $\mathcal{G}$ acts on the quotient $\mathcal{R}/\mathfrak{m}^\infty$ (this is a non-trivial statement only in the differentiable case).
   (d) In the one dimensional case and for $\mathcal{R} = \mathcal{E}$, what are the orbits of the action of $\mathcal{G}$ on $\mathcal{E}/\mathfrak{m}_{\mathcal{E}}^\infty$? Do they have distinguished representatives?
3. *(2 points)* Using Nakayama's Lemma, show that the ideal $\mathfrak{m}_{\mathcal{E}}^\infty$ in $\mathcal{E}$ is not finitely generated.

To be handed in until Thursday, 21st May 2020, (by email to `valeria.bertini@mathematik.tu-chemnitz.de`**)**