Exercises Singularity Theory

1. *(3 points)* Recall that a (commutative) ring R (with a unit element 1) is called a *local ring* iff it has exactly one maximal ideal $\mathbf{m} \subset R$.

   (a) Show that if $(R, \mathbf{m})$ is local, then for any $x \in \mathbf{m}$, the element $1 + x$ is a unit in R.

   (b) Let R be a local ring and $I \subset R$ any ideal. Show that the factor ring R/I is also local.

   (c) Let $\mathbb{R}[[x_1, \ldots, x_n]]$ be the local ring of formal power series over $\mathbb{R}$. Give an explicit expression for the inverse of $1 + x$ for $x \in \mathbf{m}$.

   (d) Show that the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$ is not local.

   (e) For any local ring, define $k := R/\mathbf{m}$. Then k is a field, called the residue class field of $(R, \mathbf{m})$. Show that the residue class field of $\mathcal{E}$ is isomorphic to $\mathbb{R}$.

   (f) Let $(R, \mathbf{m})$ be local and define by

   $$H_R(d) := \dim_k(\mathbf{m}^d/\mathbf{m}^{d+1})$$

   the Hilbert function of the local ring R. Calculate the Hilbert function for the following local rings

   i. $R = \mathcal{E}_n$, $R = \mathbb{R}[[x_1, \ldots, x_n]]$,
   ii. $R = \mathbb{R}[[x, y]]/(xy)$,
   iii. $R = \mathbb{R}[[x, y]]/(x^2 - y^3)$.

2. *(3 Punkte)*

   (a) Show that the formal power series ring $\mathbb{K}[[x_1, \ldots, x_n]] := \left\{ \sum_{\underline{\nu} \in \mathbb{N}^n} a_{\underline{\nu}} \cdot x^{\underline{\nu}} \,|\, a_{\underline{\nu}} \in \mathbb{R} \right\}$ is a $\mathbb{K}$-algebra (using the operations $+$ and $\cdot$ indicated in the lecture). Show moreover that $\mathbf{m} := \left\{ \sum_{\underline{\nu} \in \mathbb{N}^n} a_{\underline{\nu}} \cdot x^{\underline{\nu}} \,|\, a_{\underline{\nu}} \in \mathbb{R}, a_{\underline{0}} = 0 \right\}$ is a maximales ideal in $\mathbb{K}[[x_1, \ldots, x_n]]$ ist.

   (b) Show that the Taylor development map $T : \mathcal{R}_n \to \mathbb{K}[[x_1, \ldots, x_n]]$ satisfies $T([f] \cdot [g]) = T([f]) \cdot T([g])$. Show in particular (and use it) the "generalized Leibniz rule"

   $$D^{\underline{\nu}}(f \cdot g) = \sum_{\underline{\kappa} + \underline{\lambda} = \underline{\nu}} \binom{\underline{\nu}}{\underline{\kappa}} D^{\underline{\kappa}} f \cdot D^{\underline{\lambda}} g$$

   Hint: Use induction over $|\underline{\nu}|$.

3. *(2 points)* Consider the local ring $(\mathcal{E}_n, \mathbf{m})$ and let $\Psi := (\Psi_1, \ldots, \Psi_n) \in \mathbf{m}^{\oplus n} \subset (\mathcal{E}_n)^n = \mathcal{E}_{n,n}$ (caution: $\mathbf{m}^{\oplus n}$ denotes the direct sum $\mathbf{m} \oplus \ldots \oplus \mathbf{m}$, do not confuse this with the n-th power $\mathbf{m}^n$ of $\mathbf{m}$).

   (a) Show that the substitution map (also called pull-back or inverse image)

   $$\begin{array}{rcl} \Psi^* : R & \longrightarrow & R \\ f & \longmapsto & f \circ \Psi \end{array}$$

   is an algebra homomorphism preserving the identity. Show further that $\Psi^*(\mathbf{m}^k) \subset \mathbf{m}^k$.

   (b) Deduce from (a) that Ψ induces linear maps

   $$(\Psi^*)_k : \mathbf{m}^k/\mathbf{m}^{k+1} \longrightarrow \mathbf{m}^k/\mathbf{m}^{k+1}.$$

   Show that Ψ is an automorphism iff $(\Psi^*)_1$ is invertible.

To be handed in until Thursday, 14th May 2020, (by email to `valeria.bertini@mathematik.tu-chemnitz.de`)