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Exercises Singularity Theory

1. *(3 points)*

   Consider the space $\mathcal{T}(n, \mathbb{R})$ of upper triangular matrices and the space of $\mathrm{Sym}(n, \mathbb{R})$ of symmetric matrices.

   (a) Show that both are $\frac{1}{2}n(n+1)$-dimensional real vector spaces.

   (b) Let D_0 be a fixed diagonal matrix and $\alpha : \mathcal{T}(n, \mathbb{R}) \to \mathrm{Sym}(n, \mathbb{R})$ be given as $\alpha(T) := T^{tr} \cdot D_0 \cdot T$. Show that α is a local diffeomorphism at $\mathrm{Id} \in \mathcal{T}(n, \mathbb{R})$ in the following way.

   i. Show that α is smooth.

   ii. Show that $D\alpha(\mathrm{Id})$ is the linear map $\mathcal{T}(n, \mathbb{R}) \to \mathrm{Sym}(n, \mathbb{R})$ defined by $T \mapsto T^{tr} \cdot D_0 + D_0 \cdot T$.

   iii. Show that $D\alpha(\mathrm{Id})$ is injective (Hint: use the fact that T is upper triangular) and hence also surjective.

2. *(4 points)*

   Construct a local diffeomorphism at $0 \in \mathbb{R}^2$ which transforms the germ

   $$f = x^2 + y^2 + x^2 y + xy^2 \in \mathcal{E}_{\mathbb{R}^2,0}$$

   into the germ

   $$f = x^2 + y^2 \in \mathcal{E}_{\mathbb{R}^2,0}.$$

   Hint: Use the proof of the Morse lemma.

3. *(2 points)*

   (a) Let $A, B \subset \mathbb{R}^n$ open sets and (A, p) resp. (B, p) germs of sets. Show that the intersection $(A, p) \cap (B, p)$ is a well defined germ of a set.

   (b) Show analogously that the product of germs $(A, p) \times (B, q)$ is well-defined.

   (c) Show that a map germ $[f]_p = (f, p)$ has a well-defined set germ (A, p) on which it is defined.