Technische Universität Chemnitz, Fakultät für Mathematik
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Sheet 2
17.04.2020

Exercises Singularity Theory

1. *(1 point)* Let $U \subset \mathbb{R}^n$ be open, $0 \in U$ and $f : U \to \mathbb{R}$ smooth, and let $\psi : U \to U$ be a local diffeomorphism. Let moreover 0 be a critical point of f. Let $g : U \to \mathbb{R}$ be give as in the lectures by $g(y) := f(\psi(y))$. Show that $D^2g(0)$ und $D^2f(0)$ have equal rank and index.

2. *(1+2 points)* Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^2 - y^2 - xy^2 + x^2y$.
   (a) What are the rank and the index of f (i.e., the rank and the index of $D^2f(0)$) ?
   (b) Show that there is a local diffeomorphism ϕ at 0 in $\mathbb{R}^2$ which transforms f into the function $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = x^2 - y^2$.

3. *(3 points)*
   Let $U := \left\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1\right\}$ und $f : U \to \mathbb{R}$ be defined by $f(x, y) := \sqrt{1 - x^2 - y^2}$.
   (a) What are the critical points of f?
   (b) Let (x_0, y_0) be any point in $U \backslash \mathrm{Crit}(f)$. Construct a local diffeomorphism near (x_0, y_0), which transforms f into a linear function.

To be handed in until Thursday, 23th April 2020, (by email to `valeria.bertini@mathematik.tu-chemnitz.de`)