Exercises Singularity Theory

1. *(1 point)*

   Show that the following smooth functions (the so-called "elementary catastrophes") have isolated critical points at 0 in $\mathbb{R}$ resp. in $\mathbb{R}^2$ and no other critical points.

   $$x^3, x^4, x^5, x^6, x^3+y^3, x^3-xy^2, x^2y+y^4$$

2. *(4 Punkte)* A reminder from linear algebra:

   Let V be a real vector space of dimension n and $h: V \times V \to \mathbb{R}$ a symmetric bilinear map. For a given basis $(v_1, \ldots, v_n)$ of V, the map h determines a symmetric matrix $H \in \mathrm{Sym}(n, \mathbb{R}) \subset \mathrm{Mat}(n \times n, \mathbb{R})$ by putting $H_{ij} := h(v_i, v_j)$.

   The aim of the exercise is to show that there is always a basis $\underline{v}$, such that the matrix H has a particular simple shape. This is sometimes called "Sylvester's law of inertia" (Sylvesterscher Trägheitssatz).

   First we recall some useful notions needed for the statement and the proof.

   - The radical of h is defined as $\mathrm{Rad}(h) := \{v \in V \mid h(v,w) = 0 \ \forall w \in V\}$.
   - h is called non-degenerate, if $\mathrm{Rad}(h) = \{0\}$.
   - h is called positiv definite resp. negativ definite, if we have $h(v,v) > 0$ resp. $h(v,v) < 0$ for all $v \in V \backslash \{0\}$.

   Show Sylvester's law of inertia using the following hints.

   (a) *(1 point)* Let U be a subspace of V such that we have $V = \mathrm{Rad}(h) \oplus U$. Show that the restriction of h to U is non-degenerate.

   (b) *(1 point)* Let h be non-degenerate and $w \in V$ such that $h(w,w) \neq 0$. Let $W := (\mathbb{R}w)^{\perp}$ be the orthogonal complement of the space generated by w, i.e.

   $$W := \{v \in V \mid h(v,w) = 0\}.$$

   Show that the restriction of h to W is non-degenerate.

   (c) *(2 points)* Show by induction on n and using statements (a) and (b) that there is a basis $\underline{v} := (v_1, \ldots, v_n)$ of V such that the matrix H representing h with respect to $\underline{v}$ is diagonal of the form $\mathrm{diag}(\lambda_1, \ldots, \lambda_k, \epsilon_1, \ldots, \epsilon_l, \nu_1, \ldots, \nu_r)$, and such that $\lambda_1 = \ldots = \lambda_k = 1$, $\epsilon_1 = \ldots = \epsilon_l = -1$ und $\nu_1 = \ldots = \nu_r = 0$.

   Show moreover that the numbers k, l und r are independent of the choice of the basis. They are called Sylvester's invariants (of h), in particular, the number $n-r$ is called the rank of h, r is called the corank of h and l is called the index of h.

3. *(2 points)* Let $g \in \mathbb{R}[x]$ be a polynomial, $p \in \mathbb{N}$ an let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) := y^p - g(x)$.

   (a) Show that $\mathrm{Crit}(f) \cap f^{-1}(0) = \{(x,0) \mid g \text{ has multiple roots at } x\}$.

   (b) Draw the set $f^{-1}(0) \subset \mathbb{R}^2$ in the following cases

   i. $p = 1, 2$ und $g = x^2 - 1$,

   ii. $p = 2$ und $g = x^2 \cdot (1-x^2)^3$.

To be handed in until Wednesday, 15th April 2020