<u>recall</u>: Theorem 3.6.: $f \in \mathcal{R}_n$, $k \in \mathbb{N}$ s.t. $m_{\mathcal{R}_n}^{k+1} \subset m_{\mathcal{R}_n}^2 \cdot J_f$

$\Rightarrow f$ is k-determined

---

examples for calculation of μ and determinacy:

1.) $\underline{f = x^k} \in \mathcal{R}_1$, $k > 0 \Rightarrow J_f = (\partial_x f) = (k \cdot x^{k-1})$

$= (x^{k-1}) \Rightarrow \mathcal{R}_1 / J_f \cong \mathbb{K}[x] / (x^{k-1})$

$\cong \mathbb{K} \cdot \bar{1} \oplus \mathbb{K} \bar{x} \oplus \ldots \oplus \mathbb{K} \overline{x^{k-2}}$

$\uparrow$

<u>as $\mathbb{K}$-vector spaces</u>

$\Rightarrow \mu(f) = \dim_{\mathbb{K}} (\mathcal{R}_1 / J_f) = k - 1$

$\overset{3.7.}{\Longrightarrow} f$ is k-determined

f is <u>not</u> $k-1$-determined (otherwise

$f \underset{R}{\sim} 0$ ↯), hence determinacy$(f) = k$.

2.) $f = x^2 - y^2 \in \mathcal{R}_2 \rightsquigarrow J_f = (2x, -2y) = (x, y) = m_{\mathcal{R}_2}$

$\mathcal{R}_2 / J_f = \mathcal{R}_2 / m \cong \mathbb{K} \bar{1} \Rightarrow \mu(1) = 1$

<u>analogous to 1.)</u>

f is 2-determined (not 1 determined)

$\Longrightarrow$ determinacy $= 1$

generalization:

<u>Lemma 3.9.:</u> Let $f \in m_{\mathcal{R}_n}$, then

a) $\mu(f) = 0 \iff f \notin m_{\mathcal{R}_n}^2$

$\left(\iff 0 \text{ is not a critical point of } f \right)$

b) supp.: $f \in m_{\mathcal{R}_n}^2$: $\mu(f) = 1 \iff$

f has a non-deg. critical point at 0.

<u>Pf:</u> a) $\mu(f) = 0 \iff J_f = \mathcal{R}_n \iff 1 \in J_f$

$\iff \exists i \in \{1, \dots, n\} : \partial_{x_i} f \notin m_{\mathcal{R}_n} \iff f \notin m_{\mathcal{R}_n}^2$

b) „$\Leftarrow$": Morse lemma:

$\mathcal{R}=\mathcal{E}_n$: $f \underset{\mathcal{R}}{\sim} q := x_1^2+\dots+x_k^2 - x_{k+1}^2 - \dots - x_n^2$

$n-k = \operatorname{Index} D^2 f(0)$

$\mathcal{R}=\mathcal{O}_n$: $f \underset{\mathcal{R}}{\sim} q := x_1^2+\dots+x_n^2$

notice: $J_q = m_{\mathcal{R}_n}$

$\Rightarrow \mathcal{R}_n / J_f \cong \mathcal{R}_n / J_q = \mathcal{R}_n / m_{\mathcal{R}_n} \Rightarrow \mu(f) = 1$

„$\Rightarrow$": $\underline{\mu(f)=1} \Leftrightarrow J_f = m_{\mathcal{R}_n}$. Since

$f \in m_{\mathcal{R}_n}^2 \Rightarrow f \sim q + S$, where $q = \sum_{i=1}^{n} a_i x_i^2$

$a_i \in \{0,1,-1\}$, $S \in m_{\mathcal{R}_n}^3$

to show: $a_i \neq 0 \ \forall i \in \{1,\dots,n\}$

equivalently: show, that $\mu(g)=1$

from $f \sim g + S \Rightarrow J_f \subset J_g + J_S \underset{\Uparrow}{\subset} J_g + m^2$

$S \in m^3$

$\Downarrow$

$\partial_{x_i} S \in m^2$

Since $\mu(f)=1 \Rightarrow m \subset J_f \Rightarrow$

$\underline{m} \subset J_f \subset \underline{J_g + m^2} \xRightarrow{NAK} m \subset J_g$

obviously: $J_g \subset m \Rightarrow m = J_g$

$\Rightarrow \mu(g) = 1$ $\square$

3.) $f = \underline{x^2} \pm y^k \in R_2 \Rightarrow J_f = (x, y^{k-1})$

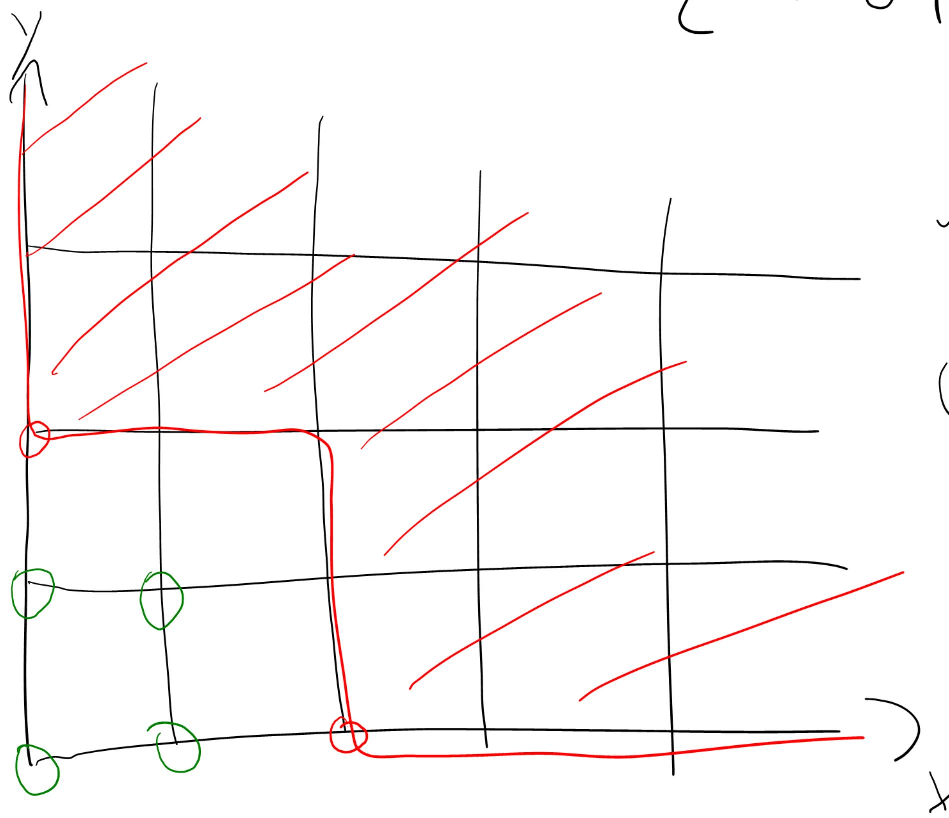
$\Rightarrow R_2 / J_f \cong \mathbb{K}\bar{1} \oplus \mathbb{K}\bar{y} \oplus \ldots \oplus \mathbb{K}\overline{y^{k-2}}$

$\Rightarrow \mu(f) = k-1 \overset{3.7}{\Longrightarrow} f$ is k-det.

f is not $k-1$-det. since then we would have $f \underset{R}{\sim} x^2 \in R_2$ ↯ $\mu(x^2 \in R_2) = \infty$

$\Rightarrow$ determinancy $= k$

4.) $f = x^3 + y^3 \in R_2$, $J_f = (x^2, y^2)$



$R_2 / J_f \cong \mathbb{K}\bar{1} \oplus \mathbb{K}\bar{x} \oplus \mathbb{K}\bar{y} \oplus \mathbb{K}\overline{xy}$

$\Rightarrow \mu(f) = 4$

$\overset{3.7}{\Longrightarrow} f$ is 5-determined

Determinancy: f not 2-determined (then would have $f \sim 0$ ↯)

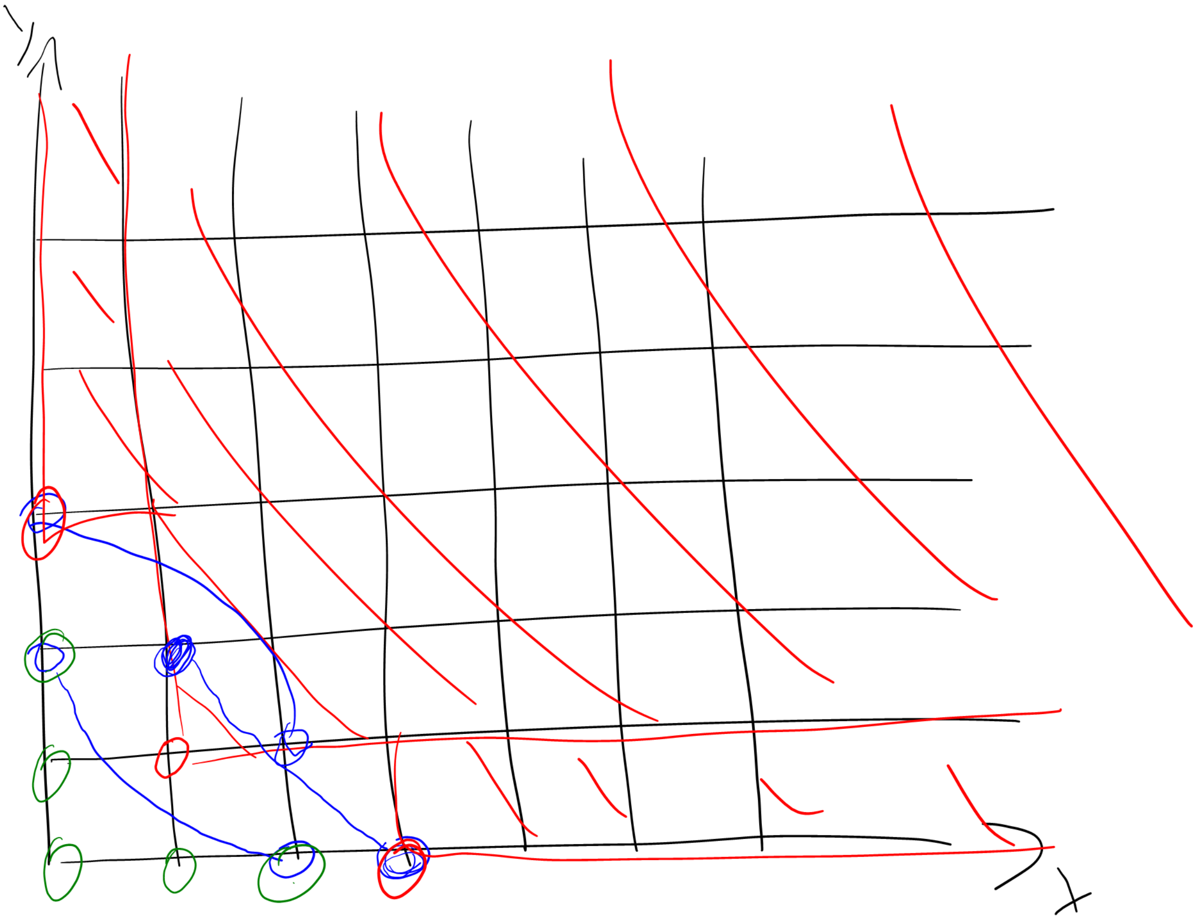
f 3-det? / f 4-det.?

$m \cdot J_f = (x,y) \cdot (x^2, y^2) = (x^3, xy^2, yx^2, y^3)$

$= m^3$, in part. $m^3 \subset m J_f \Rightarrow m^4 \subset m^2 J_f$

$\overset{3.6.}{\Longrightarrow}$ f is 3-det. (and then also 4-det.), hence, determinacy $= 3$.

5.) $f = x^3 - xy^2 \in R_2$, $J_f = (3x^2 - y^2, xy)$



we have: $3x^3 - xy^2,\ 3x^2y - y^3 \in J_f$

since $xy \in J_f \Rightarrow xy^2, x^2y \in J_f$

$\Rightarrow 3x^3, -y^3 \in J_f \Rightarrow x^3, y^3 \in J_f$

$(x^3, xy, y^3) \subset J_f \Rightarrow$

$$R_2/J_f \simeq \frac{(\mathbb{K}\bar{1} \oplus \mathbb{K}\bar{x} \oplus \mathbb{K}x^2 \oplus \mathbb{K}y \oplus \mathbb{K}y^2)}{\mathbb{K}(3x^2 - y^2)}$$

$\simeq \mathbb{K}\bar{1} \oplus \mathbb{K}\bar{x} \oplus \mathbb{K}\overline{x^2} \oplus \mathbb{K}\bar{y} \Rightarrow \mu(f) = 4$

$\Rightarrow f$ is 5-det., not 2-det

<u>determinancy:</u> $m \cdot J_f = (x,y) \cdot (3x^2 - y^2, xy)$

$= (3x^3 - xy^2, 3x^2y - y^3, x^2y, xy^2)$

$= (3x^3, -y^3, x^2y, xy^2) = m^3$

$\Rightarrow m^3 \subset m J_f \Rightarrow m^4 \subset m^2 J_f \overset{3.6.}{\Longrightarrow} f$ 3-det.

determinacy $= 3$.

6.) $f = x^4 + y^4 \in R_2$, $J_f = (x^3, y^3)$

$\Rightarrow \mu(f) = 9 \overset{3.7.}{\Longrightarrow} f$ is 10-det

(clearly f not 3-det.):

$m \cdot J_f = (x, y) \cdot (x^3, y^3) = (x^4, xy^3, y^4, yx^3)$

notice: $x^2y^2 \notin m \cdot J_f \Rightarrow m^4 \not\subset m J_f$

BUT: $m^2 J_f = (x^2, xy, y^2) \cdot (x^3, y^3) =$

$(x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5) = m^5$

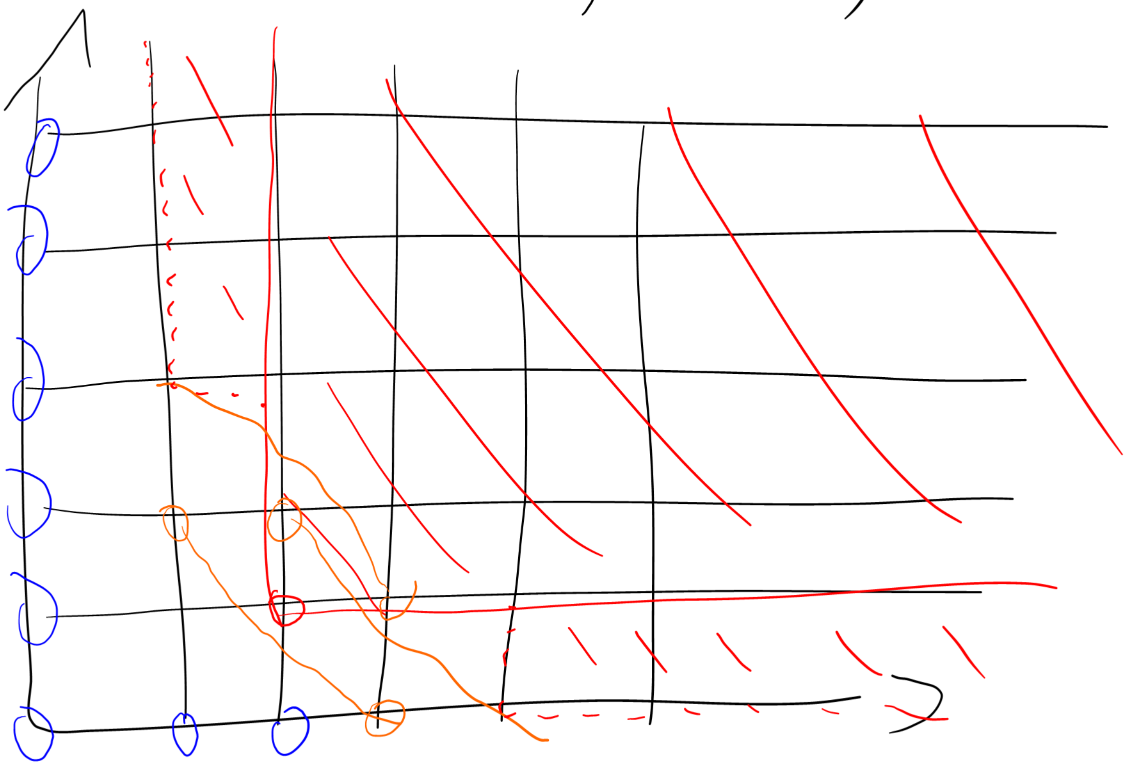
$\Rightarrow m^5 \subset m^2 J_f \Rightarrow f$ is 4-det $\Rightarrow$ determinacy $= 4$

f is k-det., but $m^k \not\subset m J_f$

(but $m^{k+1} \subset m^2 J_f$, and of course $m^{k+1} \subset m J_f$)

7.) $f = x^2 \cdot (x^2 - y^2) \in R_2$

$J_f = (4x^3 - 2xy^2, x^2 y)$

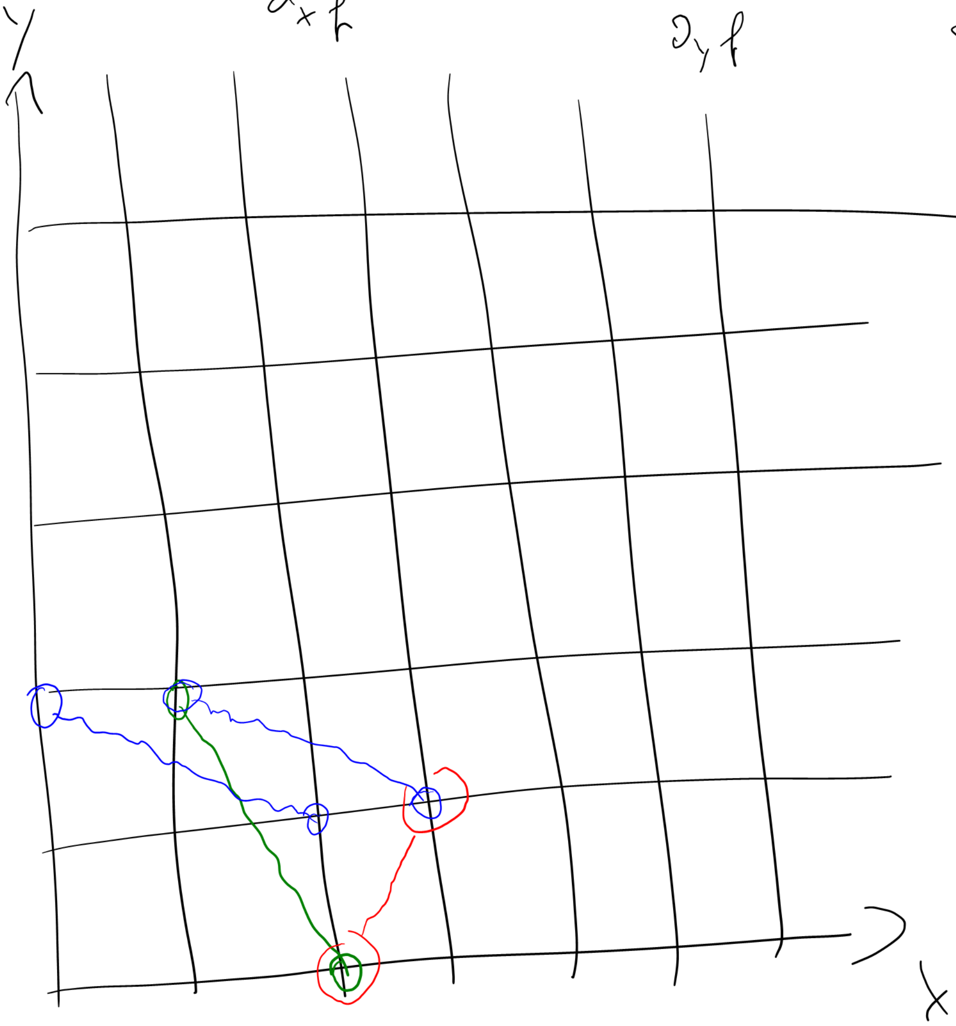


we see: $y^k \notin J_f \quad \forall k \in \mathbb{N}_0 \Rightarrow \mu(f) = \dim_{\mathbb{K}} R_2 / J_f = \infty$

later: then f is not finitely determined!

8.) $f = x^3 + y^3 + x^2y^2 \Longrightarrow$

$$J_f = \Big(\underbrace{3x^2 + 2xy^2}_{\partial_x f},\ \underbrace{3y^2 + 2x^2y}_{\partial_y f}\Big) \Longleftarrow \text{degree in } x\ y$$



we have: $x \cdot \partial_y f = 3xy^2 + 2x^3y \in J_f$

$$\Rightarrow J_f \ni \partial_x f - \frac{2}{3} \cdot x\partial_y f = 3x^2 + 2xy^2 - \frac{2}{3}\left(3xy^2 + 2x^3y\right)$$

$$= 3x^2 - \frac{4}{3}x^3y = 3x^2\left(1 - \frac{4}{9}xy\right)$$

Notice: $-\frac{4}{9}x+y \in m_{R_2} \Rightarrow 1-\frac{4}{9}x+y \in 1+m_{R_2}$

$\Rightarrow 1-\frac{4}{9}x+y \in R_2^* \quad \left(=\{g \in R_n \mid \exists h : g\cdot h = 1\}\right)$

<u>clear:</u> R ring, $c \in R^*$, $h \in R$, $I \subset R$

then: $h \in I \Leftrightarrow c\cdot h \in I$

So: $3x^2 \in J_f \Leftrightarrow x^2 \in J_f$

by symmetry of J_f in $x,y \Rightarrow y^2 \in J_f$

$\Rightarrow J_f = (x^2, y^2) \Rightarrow \mu(f) = 4$

$m \cdot J_f = (x,y)\cdot(x^2,y^2) = (x^3, xy^2, x^2y, y^3) = m^3$

$\Rightarrow m^4 \subset m^2 J_f \Rightarrow f$ is 3-det. $\Rightarrow f \underset{R}{\sim} x^3+y^3$

determinacy $= 3$.

Theorem 3.10: Supp. that $f \in R_n$ is k-determined, then $m^{k+1} \subset m J_f$.

summarizing:

$$\begin{tikzcd}
m^k \subset mJ_f \arrow[r, Rightarrow, "\text{trivial}"] \arrow[rrr, bend right, Rightarrow, "\text{trivial}"'] & m^{k+1} \subset m^2 J_f \arrow[r, Rightarrow, "3.6."] \arrow[rr, bend right, Rightarrow, "\text{trivial}"'] & f \text{ } k\text{-det.} \arrow[r, Rightarrow, "3.10"] & m^{k+1} \subset mJ_f
\end{tikzcd}$$

no full proof of 3.10 here, but one making a simplifying assumption: f k-det.

let $S \in m^{k+1} \Longrightarrow \forall t \in \mathbb{K} : f + t \cdot S \underset{R}{\sim} f$

i.e. $\exists \varphi(x,t) : f(\varphi(x,t)) = f(x) + t \cdot S(x)$

$\varphi(x,0) = x$

<u>assumption</u>: $\varphi \in R_{n+1}^{\oplus n}$, i.e. φ is infinitely many times differentiable resp. holomorphic <u>even in t</u>. This is true, but hard to show. Since $f(\varphi(x,t)) = f(x) + t \cdot S(x)$

$$\overset{\partial_t(-)}{\Longrightarrow} \sum_{i=1}^{n} (\partial_{x_i} f)(\varphi(x,t)) \cdot \frac{\partial \varphi_i}{\partial t}(x,t) = S(x)$$

Put $t=0$: $\boxed{\sum_{i=1}^{n} (\partial_{x_i} f)(\varphi(x,0)) \cdot \partial_t \varphi_i(x,0) = S(x)}$

Now: $\varphi(x,0) = x \Rightarrow (\partial_t \varphi_i)(x,0) \in m_{R_n}$

$$\Rightarrow S \in m \cdot J_f \Longrightarrow m^{k+1} \subset m \cdot J$$

$\square$

<u>Corollary 3.11:</u> 1.) Let k be minimal s.t. $m^k \subset m \cdot J_f \Rightarrow$ determinacy$(f) \in \{k, k-1\}$

2.) $f \in R_n$, $\mu(f) = \infty \Rightarrow f$ is not finitely determined

<u>Proof:</u> 1.) $m^k \subset m \cdot J_f \Rightarrow m^{k+1} \subset m^2 J_f \overset{3.6}{\Rightarrow} f$ is k-det. $\Rightarrow$ determinacy $\leq k$

If f is $k-2$-det $\xrightarrow{3.10} m^{k-1} \subset m \cdot J_f$ ↯

k was minimal with $m^k \subset m J_f$! $\Rightarrow f$ is not $k-2$-det $\Rightarrow$ determinacy $\in \{k, k-1\}$

2.) f fin. det. $\xrightarrow{3.10} \exists k : m^{k+1} \subset m J_f \subset J_f$

$R_n / m^{k+1} \twoheadrightarrow R_n / J_f$, $\mu(f) = \dim R_n / J_f) \leq \dim R_n / m^{k+1} < \infty$ □

$\boxed{\Rightarrow \text{ex. 7.: } f = x^2(x^2-y^2) \text{ not fin. det.}}$