<u>Theorem 3.6:</u> Let $f \in \mathcal{R}_n$, let $k \in \mathbb{N}$ be such that

$$m_{\mathcal{R}_n}^{k+1} \subset m_{\mathcal{R}_n}^{2} J_f.$$ Then f is k-determined.

<u>Proof:</u> Given $g \in \mathcal{R}_n$ s.t. $\underline{j^k f = j^k g}$

then we need to show that: $f \underset{\mathcal{R}}{\sim} g$. Pf in 7 steps.

1.) We have $\underline{f = g + S}$ with $S \in m^{k+1}$, hence

$$\partial_{x_i} f \in \partial_{x_i} g + m^k \ \forall i \in \{1, \dots, n\} \Longrightarrow J_f \subset J_g + m^k$$

$$\Longrightarrow m^2 J_f \subset m^2 J_g + m^{k+2}$$: hence, by assumption

of 3.6: $\underline{m^{k+1}} \overset{!}{\subset} m^2 J_f \subset m^2 J_g + m^{k+2}$

$$\underline{= m^2 J_g + m \cdot m^{k+1}} \xrightarrow{\text{Nak}} m^{k+1} \subset m^2 J_g \subset m J_g$$

So: $\forall g: j^k f = j^k g \Longrightarrow m^{k+1} \subset m J_g$.

2.) Put $F(x_1, \dots, x_n, t) := (1-t) \cdot f(\underline{x}) + t \cdot g(\underline{x})$

then $j^kF = j^kf$ we can understand F as an element in $\underline{R_n[t]} \subset R_{n+1}$

$f(\underline{x}) = F(\underline{x}, 0)$ and $g(\underline{x}) = F(\underline{x}, 1)$, moreover, we put $\forall t \in \mathbb{K}$ $f_t(x) := F(\underline{x}, t)$ $\in R_n$. We need to show: $m^{k+1} \subset m^2 J_f \Rightarrow f$ is k determined.

<u>Claim 1</u>: It is sufficient to show that

$\forall t \in \mathbb{K}: \exists \varepsilon > 0: f_s \underset{R}{\sim} f_t \quad \forall s \in \Delta_\varepsilon(t) :=$

$$\{s \in \mathbb{K} \mid |s-t| < \varepsilon\}$$

Clear: since $[0,1] \subset \mathbb{K}$ is compact, $\forall$ coverings $[0,1] = \bigcup \Delta_\varepsilon(t)$, $\exists$ finite subset of coverings. $\Rightarrow f_0 = f \underset{R}{\sim} f_1 = g$

Claim 2.: In order to show Claim 1, it suffices: $\forall g' \in R_{n}$, $S \in m^{k+1}$ s.t. $m^{k+1} \subset m J_{g'} \Rightarrow \exists \varepsilon > 0 : g' \sim g' - t \cdot S$ $\forall t \in \Delta_{\varepsilon} := \Delta_{\varepsilon}(0)$. Clear since by 1.) of the proof we have $m^{k+1} \subset m J_{g'}$ $\forall g'$ s.t. $j^{k} f = j^{k} g'$, in particular for $g' = f_t$

3.) we need to prove: $g' \in R_n$, $\underline{m^{k+1} \subset m J_{g'}}$, $\underline{S \in} m^{k+1} \Rightarrow \exists \varepsilon > 0 : g' \sim \underline{g' - t \cdot S}$ $\forall t \in \Delta_{\varepsilon}$

Write $\underline{G(\underset{\sim}{x}, t)} := g' - t \cdot S \in R_{n+1}$

It suffices to find open nghb.

$0 \in U \subset \mathbb{K}^n$ and a $\begin{cases} C^\infty \\ hol. \end{cases}$ map

$H : U \times \Delta_\varepsilon \longrightarrow U$ s.t.:

i) $H(\underline{x}, 0) = \underline{x}$

ii) $H(0, t) = 0$

iii) $G(H(\underline{x}, t), t) = g'(x) \; (= G(x, 0))$

$\forall \underline{x} \in U$
$\forall t \in \Delta_\varepsilon$

Sufficient, since: $\forall t \in \Delta_\varepsilon$; ii) implies

$h_t := H(-, t) \in m_{R_n}^{\oplus n}$; moreover, i) implies

$D h_0 = Id_n$ and because of continuity

of $D h_t$ we have $\det (D h_t) \neq 0$

$\forall t \in \Delta_\varepsilon$ for some $\varepsilon > 0$.

Hence $\forall t \in \Delta_\varepsilon$: $h_t \in \mathcal{G}_n$

Hence iii) $\Rightarrow$ $g' - tS = G(-,t) \sim g'$ $\forall t \in \Delta_\varepsilon$

4.) in 3.) it suffices to find U and H s.t. i), ii) and iii)' hold, where iii)' $:= \partial_t$(iii)):

$$\boxed{\partial_t G(H(x,t),t) = \partial_t g'(x) = 0}$$

This holds since $\partial_t G(H(x,t),t) = 0$ is an ODE with coefficients from R_n and i), ii) are initial values.

iii)': $\partial_t G(H(\underline{x},t),t)$

$$= \boxed{\sum_{i=1}^{n} (\partial_{x_i} G)(H(\underline{x},t),t) \cdot (\partial_t H_i)(\underline{x},t) + (\partial_t G)(H(\underline{x},t),t) = 0} \quad (*)$$

where $H(\underline{x},t) = (H_1(\underline{x},t), \dots, H_n(\underline{x},t))$

so given G ($:= g' - t \cdot S$, where $m^{k+1} \subset m J_{g'}$, $S \in m^{k+1}$), we need to find H satisfying (*).

5.) <u>Claim 3:</u> From $\underline{m^{k+1} \subset m J_{g'}}$ it follows: $\exists\, \eta_1, \dots, \eta_n \in R_{n+1}$:

a) $\eta_i(0,t) = 0$

b) $S = \sum_{i=1}^{n} (\partial_{x_i} G)(x,t) \cdot \eta_i(x,t)$

Pf of claim 3: notation: If $I \subset R_n$ is ideal, then write $I \cdot R_{n+1}$ for the ideal generated by I in R_{n+1} (notice that $R_n \subset R_{n+1}$ is a subring). Recall $G(\underline{x}, t) := g' - t \cdot S$

$$S \in m_{R_n}^{k+1} \implies \boxed{\partial_{x_i} G = \partial_{x_i} g' - t \cdot \underbrace{\partial_{x_i} S}_{m_{R_n}^{k}}}$$

$$\implies \partial_{x_i} g' \in \underset{\substack{\shortparallel \\ (\partial_{x_1} G, \ldots, \partial_{x_n} G) \subset R_{n+1}}}{\tilde{J}_G} + (t) \cdot m_{R_n}^{k} \cdot R_{n+1}$$

$$J_{g'} \cdot R_{n+1} \overset{(*)}{\subset} \tilde{J}_G + (t) \cdot m_{R_n}^{k} \cdot R_{n+1}$$

Since: $m_{R_n}^{k+1} \subset m_{R_n} \cdot J_{g'} \implies m_{R_n}^{k+1} \cdot R_{n+1} \subset m_{R_n} J_{g'} \cdot R_{n+1}$

from (**) :

$$\underline{m_{R_n}^{k+1}\cdot R_{n+1} \subset m_{R_n}\cdot J_G\cdot R_{n+1}} \subset m_{R_n}\tilde{J}_G + \underbrace{(t)}_{\subset\, m_{R_{n+1}}}\cdot \underbrace{m_{R_n}^{k}\cdot R_{n+1}}_{\subset\, m_{R_n}^{k+1} R_{n+1}}$$

so $m^{k+1} \qquad \subset m_{R_n}\tilde{J}_G +$

$$\xRightarrow{\text{NAK}} m_{R_n}^{k+1}\cdot R_{n+1} \subset \overbrace{m_{R_n}\tilde{J}_G}^{\subset R_{n+1}}$$

hence: from $S \in m_{R_n}^{k+1}$ (so that we can see $S \in m_{R_n}^{k+1}\cdot R_{n+1}$) we get

$$S \in m_{R_n}\tilde{J}_G \Rightarrow \exists\, \varphi_1, \dots, \varphi_n \in R_{n+1}$$

s.t. $S = \sum \varphi_i(x,t)\cdot \partial_{x_i} G(x,t)$

and $\varphi_i \in m_{R_n}\cdot R_{n+1}$, i.e. $\varphi_i(0,t) = 0$

6.) Let $\varphi_i \in m_{\mathcal{R}_{n+1}} \cdot \mathcal{R}_{n+1} \subset \mathcal{R}_{n+1}$ be given, consider the system of ODE

$$\boxed{\left.\begin{aligned} \partial_t H_i(\underline{x},t) &= \varphi_i\big(H(x,t),t\big) \\ H(x,0) &= \underline{x} \end{aligned}\right\} \quad (\ast\ast\ast)}$$

Usual existence theorems for ODE's yield unique solution of $(\ast\ast\ast)$ on some $U \times \Delta_\varepsilon$, $0 \in U \subset \mathbb{K}^n$, since $\underbrace{H(x,0)=\underline{x}}_{\Downarrow \atop D\varphi_0(0)=\mathrm{Id}}$

so $\det D\varphi_t(0) \neq 0 \quad \forall\, t \in \Delta_\varepsilon$

7.) from 5): (recall $G(\underline{x},t) = g'(\underline{x}) - t \cdot S(\underline{x})$)

$$S = -(\partial_t G)(\underline{x},t) = -\sum_{i=1}^{n} \varphi_i(x,t) \cdot (\partial_{x_i} G)(\underline{x},t)$$

$$\forall\, (\underline{x}, t) \in U \times \Delta_\varepsilon$$

$\Longrightarrow$

$$(\partial_t G)(H(x,t),t) = \sum \varphi_i(H(x,t),t) \cdot (\partial_{x_i} G)(H(x,t),t)$$

since $h_t = H(-,t) \in \mathcal{G}_\mu \quad \forall t \in D_\varepsilon$

$$\overset{6.)}{\Longrightarrow} (\partial_t G)(H(x,t),t) = \sum (\partial_t H_i)(x,t) \cdot (\partial_{x_i} G)(x,t)$$

This is the ODE $(*)$!!!

$\square$

<u>Lemma 3.8.:</u> $f, g \in \mathcal{R}_n$; $f \underset{R}{\sim} g$

a) $\mu(f) = \mu(g)$ (i.e. $\mu(f) < \infty \iff \mu(g) < \infty$ and, if finite, they are equal)

b) f is k-det. $\Leftrightarrow g$ is k-det.

Proof: a) Let $\varphi = (\varphi_1, \dots, \varphi_n) \in \mathcal{G}_n$ with $\varphi^* f = g$

i.e. $g(x) = f(\varphi(x))$, $\mathcal{G}_n$ is group

$\Rightarrow \exists \psi \in \mathcal{G}_n : \boxed{\varphi \circ \psi = \psi \circ \varphi = \mathrm{Id}}$, $f(x) = g(\psi(x))$

$$\Rightarrow \underline{\partial_{x_i} g} = \sum_{j=1}^{n} \underline{(\partial_{x_j} f)(\varphi(x))} \cdot \frac{\partial \varphi_j}{\partial x_i}(x)$$

so $\partial_{x_i} g \in \varphi^* J_f \Rightarrow \underline{J_g \subset \varphi^* J_f}$

similarly: $\underline{J_f \subset \psi^* J_g}$ $\Rightarrow J_g = \varphi^* J_f$

since $\boxed{\varphi^* \circ \psi^* = \psi^* \circ \varphi^* = \mathrm{Id}_{R_n}}$ $\quad J_f = \psi^* J_g$

$$\Rightarrow R_n / J_f \underset{\psi^*}{\overset{\varphi^*}{\rightleftarrows}} R_n / J_g \quad (\cong)$$

hence $\dim_{\mathbb{K}}(\mathcal{R}_n/J_g) = \dim_{\mathbb{K}}(\mathcal{R}_n/J_g)$

$$\mu(g) = \mu(f)$$

b) if $\varphi \in \mathcal{G}_n \subset \mathcal{R}_n^{\oplus n} \Rightarrow \varphi^* m^k \subset m^k$

let f be k-determined (need to show: g is k-det.): let $S \in m^{k+1}$ and $g' = g + S$, show $g \underset{R}{\sim} g'$. Let $\varphi \in \mathcal{G}_n$ with $f = \varphi^* g$. Then:

$$g' \sim \varphi^* g' = \varphi^*(g+S) = \underbrace{\varphi^* g}_{f} + \varphi^* S$$

$$= f + \overbrace{\varphi^* S}^{\in m^{k+1}} \overset{f \text{ is } k\text{-det}}{\underset{R}{\sim}} f \overset{\text{ass.}}{\sim} g \Rightarrow g' \sim g \quad \square$$