Theorem 2.11: The map

$$T : \mathcal{R}_n \longrightarrow \mathbb{K}[[x_1, \dots, x_n]]$$

$$f \longmapsto \sum_{\nu \in \mathbb{N}^n} \frac{1}{\nu!} (D^\nu f)(0) \cdot x^\nu$$

is $\mathbb{K}$-algebra hom. and is

$$\begin{cases} \text{inj., not surj.} & \text{if } \mathcal{R}_n = \mathcal{O}_n \\ \underline{\text{surj}}\text{, not inj} & \text{if } \mathcal{R}_n = \mathcal{E}_n \end{cases}$$

---

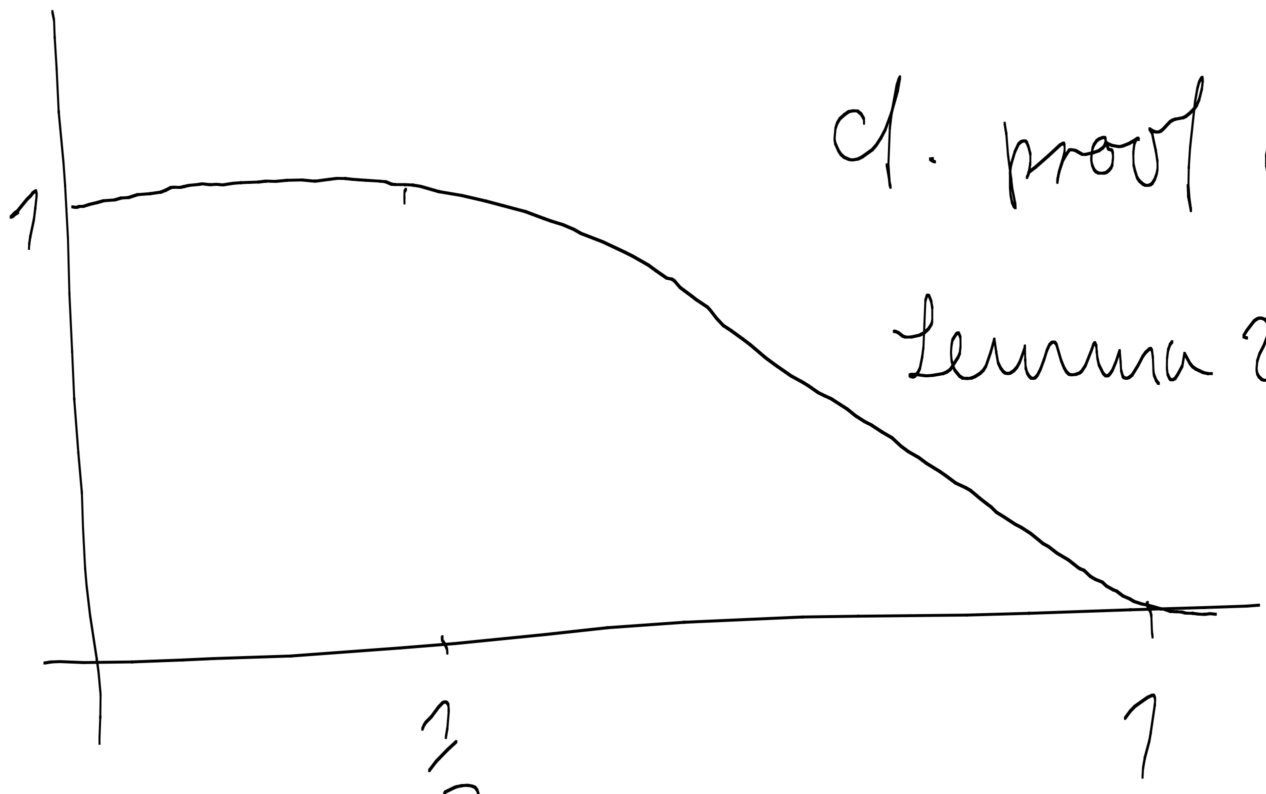
to show: $T : \mathcal{E}_n \to \mathbb{R}[[x_1 \dots x_n]]$ surjective, i.e.

$\forall \, (a_\nu)_{\nu \in \mathbb{N}^n}, \; a_\nu \in \mathbb{R} \implies \exists f \in \mathcal{E}_n$ such that

$$T(f) = \boxed{\sum_{\nu \in \mathbb{N}^n} \frac{1}{\nu!} a_\nu \cdot x^\nu}$$

<u>Def.:</u> $u : \mathbb{R}^n \to \mathbb{R}$ by

$$u(x) = \begin{cases} 0 & \forall x : |x| \geq 1 \\ \left(1 + e^{\frac{1}{\frac{1}{2} - |x|} + \frac{1}{1 + |x|}}\right)^{-1} & \forall x : \frac{1}{2} < |x| < 1 \\ 1 & \forall x : |x| \leq \frac{1}{2} \end{cases}$$



cf. proof of Lemma 2.6

$u \in \mathcal{C}^\infty(\mathbb{R}^n)$

put $\forall \nu \in \mathbb{N}^n : b_\nu := \begin{cases} 1/2 \cdot |a_\nu| & \forall |a_\nu| \geq 1 \\ 1/2 & \forall |a_\nu| < 1 \end{cases}$

$\Rightarrow |a_\nu| \cdot b_\nu \leq \frac{1}{2}$ define

$$g_\nu(x) := \underbrace{\frac{1}{\nu!} \cdot a_\nu \cdot \boxed{u\left(\frac{x}{b_\nu}\right) \cdot x^\nu}}$$

$$f(x) := \sum_{\nu \in \mathbb{N}^n} g_\nu(x)$$

<u>to show:</u> 1.) $[f] \in \mathcal{E}_n$; 2.) $(D^\lambda f)(0) = a_\lambda$

<u>Claim 1:</u> $\forall \lambda \in \mathbb{N}^n$: $\boxed{\sum_{\nu \in \mathbb{N}^n} (D^\lambda g_\nu)(x)}$ is uniformly convergent on $\mathbb{R}^n$

in particular, this holds for $\alpha = 0$

$\leadsto$ for $\sum g_\nu(x) = f(x)$

<u>Claim 2:</u> $\forall\, \underline{\nu \geq 1}$ (i.e. $\nu_i \geq 1 \;\forall i \in \{1 \dots n\}$)

$$|D^\alpha g_\nu(x)| \leq C_\alpha \cdot 2^{-\underbrace{|\nu|}_{\sum_{i=1}^n \nu_i}}$$

<u>ok:</u> $2^{-|\nu|} = 2^{-\sum_{i=1}^n \nu_i} = \prod_{i=1}^n 2^{-\nu_i}$

$$\Rightarrow \sum_{\nu \in \mathbb{N}^n} 2^{-|\nu|} = \prod_{i=1}^n \sum_{k=0}^\infty 2^{-k} = \Big(\underbrace{\sum_{k=0}^\infty 2^{-k}}_{\frac{1}{1-\frac{1}{2}} = 2}\Big)^n$$

$$= 2^n$$

Proof of Claim 2 (omitted in lecture):

We have $\forall \underline{\lambda} < \underline{\nu}$:

$$\left|D^{\underline{\lambda}} g_{\underline{\nu}}(x)\right| = \frac{1}{\underline{\nu}!} |a_{\underline{\nu}}| \cdot \left| D^{\underline{\lambda}}\left(u\left(\frac{x}{b_{\underline{\nu}}}\right) \cdot x^{\underline{\nu}}\right)\right|$$

$$= \frac{1}{\underline{\nu}!} |a_{\underline{\nu}}| \cdot \left| \sum_{0 \le \underline{\kappa} \le \underline{\lambda}} \binom{\underline{\lambda}}{\underline{\kappa}} \cdot D^{\underline{\lambda}-\underline{\kappa}} u\left(\frac{x}{b_{\underline{\nu}}}\right) \cdot \frac{\underline{\nu}!}{(\underline{\nu}-\underline{\kappa})!} x^{\underline{\nu}-\underline{\kappa}} \right|$$

by induction

$$D^{\underline{\lambda}-\underline{\kappa}} u\left(\frac{x}{b_{\underline{\nu}}}\right) = \left(D^{\underline{\lambda}-\underline{\kappa}} u\right)\left(\frac{x}{b_{\underline{\nu}}}\right) \cdot \frac{1}{b_{\underline{\nu}}^{|\underline{\lambda}-\underline{\kappa}|}} \quad (*)$$

moreover: $\left. \begin{aligned} |x^{\underline{\nu}-\underline{\kappa}}| &= \left|\prod_{i=1}^{n} x_i^{\nu_i-\kappa_i}\right| = \prod_{i=1}^{n} |x_i|^{\nu_i-\kappa_i} \\ &\overset{|x_i| \le |x| = (\sum x_i^2)^{\frac{1}{2}}}{\le} \prod_{i=1}^{n} |x|^{\nu_i-\kappa_i} = |x|^{\sum_i \nu_i-\kappa_i} = |x|^{|\underline{\nu}-\underline{\kappa}|} \end{aligned} \right\} (**)$

by definition $u\left(\frac{x}{b_{\underline{\nu}}}\right) = 0 \ \forall |x| \ge b_{\underline{\nu}}$, hence

$D^{\underline{\lambda}-\underline{\kappa}} u\left(\frac{x}{b_{\underline{\nu}}}\right) = 0 \ \forall |x| \ge b_{\underline{\nu}}$, so we can assume that

$$|x|^{|\nu-\kappa|} \underset{(***)}{\le} |b_\nu|^{|\nu-\kappa|}$$

Put $K_\lambda = \max\limits_{\nu, x}\left\{ |D^{\vartheta} u(x)| \;\middle|\; 0 \le \vartheta \le \lambda, x \in \mathbb{R}^n \right\}$, notice:

K_λ exists, since $u(x) = 0 \quad \forall |x| \ge 1$.

Summarizing, we have: $\forall \nu \ge \lambda$ $\left|\left(D^\lambda g_\nu\right)(x)\right| =$

$$\frac{1}{\nu!} |a_\nu| \cdot \left| \sum_{0 \le \kappa \le \lambda} \binom{\lambda}{\kappa} \cdot \underbrace{D^{\lambda-\kappa} u\left(\frac{x}{b_\nu}\right)}_{(*)} \cdot \frac{\nu!}{(\nu-\kappa)!} \underbrace{x^{\nu-\kappa}}_{(**)+(***)} \right|$$

$$\le |a_\nu| \cdot \lambda! \cdot K_\lambda \cdot \sum_{0 \le \kappa \le \lambda} \frac{1}{b_\nu}^{|\lambda-\kappa|} \cdot b_\nu^{|\nu-\kappa|}$$

$$\frac{1}{\kappa!\,(\lambda-\kappa)!\,(\nu-\kappa)!} \le 1$$

$$= |a_\nu| \cdot \lambda! \cdot K_\lambda \cdot \sum_{0 \le \kappa \le \lambda} b_\nu^{|\nu-\lambda|}$$

Recall: $b_\nu := \begin{cases} \frac{1}{2 \cdot |a_\nu|} & \forall |a_\nu| > 1 \\ \frac{1}{2} & \forall |a_\nu| < 1 \end{cases}$ d.h. $|a_\nu| \cdot b_\nu \le \frac{1}{2}$

and:

$$|D^{\underline{\lambda}} g_{\underline{\nu}}(x)| \le |a_{\underline{\nu}}| \cdot \theta_{\underline{\nu}} \cdot \lambda! \cdot K_{\underline{\lambda}} \cdot \sum_{0 \le \underline{\kappa} \le \underline{\lambda}} \theta_{\underline{\nu}}^{|\underline{\nu} - \underline{\lambda}| - 1}$$

Notice: $\theta_{\underline{\nu}}^{|\underline{\nu}-\underline{\lambda}|-1} \le \left(\frac{1}{2}\right)^{|\underline{\nu}-\underline{\lambda}|-1} = 2^{|\underline{\lambda}|+1} \cdot 2^{-|\underline{\nu}|}$, hence, for $\underline{\nu} > \underline{\lambda}$

$$D^{\underline{\lambda}} g_{\underline{\nu}}(x) \le \underbrace{\left(\lambda! \cdot K_{\underline{\lambda}} \cdot \sum_{0 \le \underline{\kappa} \le \underline{\lambda}} 2^{|\underline{\lambda}|+1}\right)}_{=:\, C_{\underline{\lambda}}} \cdot 2^{-|\underline{\nu}|} \le C_{\underline{\lambda}} \cdot 2^{-|\underline{\nu}|}$$

This shows claim 1

To show Claim 1, it suffices to show uniform convergence of

$$\sum_{\nu \geq 1} D^{\lambda} g_\nu(x) \quad \text{for all } \lambda \in \mathbb{N}^n$$

we have: $\sum_{\nu \geq 1} |D^{\lambda} g_\nu(x)| \leq \sum_{\nu \geq 1} c_\lambda \cdot 2^{-|\nu|}$

$$\leq c_\lambda \cdot 2^n \quad \checkmark$$

$\leadsto$ $f(x)$ function on $\mathbb{R}^n$ AND

(analysis): $(D^{\lambda} f)(x) = \sum_{\nu \in \mathbb{N}^n} D^{\lambda} g_\nu(x)$

$\Longrightarrow f \in C^\infty(\mathbb{R}^n) \Rightarrow [f]_0 \in \mathcal{E}_n$

since $(D^{\lambda} u)(0) = 0 \quad \forall \lambda \in \mathbb{N}^n \setminus \{0\}$

$$\Rightarrow (D^\lambda g_\nu)(0) = \frac{1}{\nu!} a_\nu \cdot \sum_{0 \le \kappa \le \lambda} \binom{\lambda}{\kappa} D^{\lambda-\kappa} u(0) \cdot \frac{\nu!}{(\nu-\kappa)!} x^{\nu-\kappa}(0)$$

$$\underline{D^{\lambda-\kappa} u(0) \cdot x^{\nu-\kappa}(0)} = \begin{cases} 0 & \forall \lambda \neq \kappa \\ 0 & \forall \nu \neq \kappa \end{cases}$$

$$\Rightarrow (D^\lambda g_\nu)(0) = \begin{cases} 0 & \forall \lambda \neq \nu \\ \frac{1}{\nu!} a_\nu \end{cases}$$

$$\Rightarrow T(f)(x) = \sum_{\lambda \in \mathbb{N}^n} \frac{1}{\lambda!} (D^\lambda f)(0)\, x^\lambda$$

$$= \sum_{\lambda} \frac{1}{\lambda!} \left(D^{\lambda} \sum_{\nu} g_{\nu} \right)(0)\, x^{\lambda} =$$

$$\sum_{\nu \in \mathbb{N}^n} \frac{1}{\nu!} \left(D^{\nu} g_{\nu} \right)(0)\, x^{\nu} = \sum_{\nu \in \mathbb{N}^n} \frac{1}{\nu!} a_{\nu} x^{\nu}$$

$\square$

Local rings

Def. 2.12: A commutative ring (with unit 1) is called local, if it has exactly one maximal ideal, denoted by m_R.

We would like to show: $\mathcal{R}_n$ and $\mathbb{K}[[x_1, \dots, x_n]]$ are local rings with max. ideals $\{[f] \in \mathcal{R} \mid f(0) = 0\}$ resp.

$$\left\{ \sum_{\nu \in \mathbb{N}^n \setminus \{0\}} a_\nu \cdot x^\nu \;\middle|\; a_\nu \in \mathbb{K} \right\}$$

Lemma 2.13: R is local iff

$$\mathfrak{M} = \{ a \in R \mid a \notin R^\times \}$$

is an ideal (and then automatically <u>the</u> maximal ideal), where

$$R^\times = \{ a \in R \mid \exists\, b \in R,\ a \cdot b = b \cdot a = 1 \}$$

is the set of units in R.

<u>Pf</u>: 1.) $\forall x \in R$: $\underline{x \in R^\times \iff (x) = R}$:

obvious: $R = (1)$ hence to show: $\underline{x \in R^\times \iff (x) = (1)}$

$\Rightarrow$: $x \in R^\times \Rightarrow \exists y \in R: xy = 1 \Rightarrow (1) \subset (x)$

but $(x) \subset (1) = R$ ✓

$\Leftarrow$: $(1) \subset (x) \Rightarrow 1 \in (x) \Rightarrow \exists y \in R: xy = 1$

$\Rightarrow x \in R^*$.

2.) $N\mathcal{U} \overset{!}{=} \bigcup_{\substack{m \text{ max.} \\ \text{ideal in } R}} m$

"$\subseteq$": use Zorn's lemma, which implies: $\forall J \subsetneq R$: $\exists$ maximal ideal $m \subset R$ s.t. $J \subset m$

$\Rightarrow \forall x \in N\mathcal{U} \Rightarrow \exists m$ max. ideal $(x) \subset m$ (since by 1.) $(x) \subsetneq R$)

$\Rightarrow x \in \bigcup_{m \text{ max}} m$

"$\supseteq$": Let $m \subset R$ be maximal, $x \in m$

to show: $x \in NU$, i.e. $x \notin R^\times$

use 1.): to show: $(x) \subsetneq R$

obvious, since $x \in m \Rightarrow (x) \subset m$

and $m \subsetneq R$ (by def. of max. ideal)

3.) <u>show:</u> R local $\Longrightarrow$ NU is ideal

Let <u>R be local with max. ideal m</u>

$$\Longrightarrow NU \overset{2.)}{=} \bigcup_{m \text{ max}} m \overset{\downarrow}{=} m$$

hence $NU = m$ is an ideal

4.) <u>show:</u> R local $\Longleftarrow$ <u>NU is ideal</u>

let $NU \subset R$ be an ideal

then it must be maximal:

Suppose we have $NU \subset I \subset R$

then $NU \subsetneq I \Rightarrow \exists x \in I,\ x \notin NU$,

hence $x \in R^* \Leftrightarrow (x) = R \xRightarrow{(x) \subset I} I = R$

hence NU is maximal.

Suppose $m \subset R$ maximal and

$m \neq NU \Rightarrow m \not\subset NU$, i.e.

$\exists x \in m,\ x \notin NU$, i.e. $x \in R^*$

$x \in R^* \Leftrightarrow (x) = R$

$x \in m \Longrightarrow m = R$ ↯ m max.

$\Rightarrow NU$ the only max. ideal $\Rightarrow R$ local $\square$

apply this to $\mathcal{R}_n$ and $\mathbb{K}[x_1, \dots, x_n]$

Lemma 2.14: (a) $[f]$ is unit $\Leftrightarrow f(0) \neq 0$

(b) $\sum a_\nu x^\nu \in (\mathbb{K}[\![x_1 \dots x_n]\!])^* \Leftrightarrow a_{\underline{0}} \neq 0$

(c) $\mathcal{R}_n$ and $(\mathbb{K}[\![x_1 \dots x_n]\!])^*$ are local with

max. ideals $m_{\mathcal{R}} = \{ [f] \mid f(0) = 0 \}$ resp.

$m_{\mathbb{K}[\![x_1, \dots, x_n]\!]} = \{ \sum_{\nu \in \mathbb{N}^n \setminus \{0\}} a_\nu x^\nu \mid a_\nu \in \mathbb{K} \}$

and the Taylor development

$T : \mathcal{R} \longrightarrow \mathbb{K}[\![x_1 \dots x_n]\!]$ is a local

map, i.e. $T(m_{\mathcal{E}}) \subset m_{\mathbb{K}[\![\underline{x}]\!]}$

<u>Proof</u>: (a) if $f(0)=0$ and if

$\exists\, [g] \in \mathcal{R} : [f]\cdot[g] = [1] \Rightarrow$ ↯

since it would yield $\underbrace{f(0)}_{\overset{\shortparallel}{0}} \cdot g(0) = 1$ ↯

this shows „$\Rightarrow$" in (a)

„$\Leftarrow$": Let $[f] \in \mathcal{R}_n$, $f(0) \neq 0 \Rightarrow$

$\exists\, 0 \in U \subset \begin{cases} \mathbb{C} \\ \mathbb{R} \end{cases}$ ngh. and a representative

$f \in \begin{cases} \mathcal{O}(U) \\ C^\infty(U) \end{cases}$ s.t. $f(x) \neq 0 \quad \forall x \in U$

put $g(x) := 1/f(x) \in \begin{cases} \mathcal{O}(U) \\ C^\infty(U) \end{cases}$

$\Rightarrow [f]\cdot[g] = 1 \Rightarrow [f] \in \mathcal{R}^*$

b) $a_0 = 0$! $\forall \sum_{k \in \mathbb{N}^n} b_k \cdot x^k \in \mathbb{K}[[x]]$

$$\Rightarrow \left(\sum_\lambda a_\lambda x^\lambda\right) \cdot \left(\sum_k b_k \cdot x^k\right) = \sum_\nu c_\nu \cdot x^\nu$$

with $c_0 = a_0 \cdot b_0 \overset{\downarrow}{=} 0$

$$\Rightarrow \sum c_\nu \cdot x^\nu \neq 1 \Rightarrow \sum a_\lambda \cdot x^\lambda \notin (\mathbb{K}[[x]])^*$$

if $a_0 \neq 0 \Rightarrow \exists \sum_k b_k \cdot x^k$ s.t.

$$\left(\sum a_\lambda x^\lambda\right) \cdot \left(\sum b_k \cdot x^k\right) = 1$$

exercise !

c) 2.13. implies that R_n and $\mathbb{K}[[x]]$

are local and obviously:

$$T(f) \in m_{\mathbb{K}[[x]]} \text{ if } f(0)=0,$$

i.e. if $f \in m_{\mathcal{R}_1}$

$\square$

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aim: show important structure theorem for modules over local rings.

<u>Def./Lemma 2.15:</u> R comm. ring with 1.

a) let M be an R-module, and let $A \subset M$ be subset, then we call

$$R \cdot A = \left\{ \sum_{i=1}^{k} r_i \cdot m_i \;\middle|\; k \in \mathbb{N}, r_i \in R, m_i \in A \right\}$$

the submodule of M generated by A. M is called finitely generated if $\exists A \subset M$, A finite s.t. $R \cdot A = M$.

b) If N and M are R-modules s.t. $N \subset M$ submodule, then M/N is also an R-module.

c) If M is R-module, $I \subset R$ ideal, then $I \cdot M := \{\sum r_i \cdot m_i \mid r_i \in I, m_i \in M\}$ $\subset M$ is an R-module (submodule of M and $M/I \cdot M$ is an R-module (as stated in c)), but even an R/I-module.

Pf: ex.

(ok: c): Let $I \subset R$ ideal let N be R-module, when is N also an R/I-module:

Answer: if $I \cdot N = \{0\}$)

Theorem 2.16. (Lemma of Nakayama). Let (R, m) be a local ring. Then

(a) Let A be a finitely generated R-module s.t. $A \subset m \cdot A$.

Then $A = \{0\}$

(b) Let A be an R-module, $B \subset A$, $C \subset A$ R-submodules s.t. B is finitely generated. Suppose that

$B \subset C + m \cdot B$. Then $B \subset C$

(c) A finitely generated R-module.

Then: A is generated by elements

$a_1, \dots, a_k \in A$.

$\boxed{\Longleftrightarrow}$ $\Rightarrow$ trivial

$\Leftarrow$ strong

The classes $[a_1], \dots, [a_k]$ in A/mA

generate A/mA as R/m-module

(i.e., since $k := R/m$ is a field,

the classes $[a_1], \dots, [a_k]$ form

a k-basis of the k-vector space

A/mA)