Def. 2.3.: i) $A, B \subset \mathbb{R}^n$, $p \in \mathbb{R}^n$: $A \sim_p B \iff$ $\exists\, p \in U$ open s.t. $A \cap U = B \cap U$. (A,p) is equ.-class of A and is called germ of A at p

ii) $Y \subset \mathbb{R}^m$ open, $f: A \to Y$, $g: B \to Y$ maps, $p \in \mathbb{R}^n$

$f \sim_p g \iff \exists\, p \in U \subset \mathbb{R}^n$ open s.t.

$f|_{A \cap U} = g|_{B \cap U}$. (f,p) or $[f]_p$ is called (map) germ of f at p

---

Lemma 2.4.: i) $A, B \subset \mathbb{R}^n$, $p \in \mathbb{R}^n$, (A,p), (B,p) germs then $(A,p) \cap (B,p) = (A \cap B, p)$ is well-defined germ

ii) analogous: $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$, $p \in \mathbb{R}^n$, $q \in \mathbb{R}^m$

$\implies (A,p) \times (B,q)$ well-defined

iii) a map germ $\underline{(f,p)}$ of a map $f: A \to Y$ at p has a well-defined set germ (A,p) as „germ of domain of definition", write $\boxed{(f,p) : (A,p) \to (Y, f(p))}$.

Lemma 2.5: Let $p \in \mathbb{R}^n$, then the set

$$\mathcal{E}_{\mathbb{R}^n,p} := \mathcal{E}_p := \left\{ [f]_p \;\middle|\; f \in C^\infty(U), \text{ for } U \text{ open ngh. of } p \text{ in } \mathbb{R}^n \right\}$$

is an $\mathbb{R}$-algebra and $m_p := \{ [f]_p \mid f(p) = 0 \} \subset \mathcal{E}_p$ is a maximal ideal. w.r.t. inclusion

Rk: - R ring, $I \subset R$ is called ideal:

1.) $(I,+) \subset (R,+)$ subgroup

2.) $\forall r \in R,\ \forall i \in I : r \cdot i \in I$

- $[f]_p$ then $f(p)$ is well-defined but $f(q)$ is not $\forall q \neq p$.

Pf: Let $[f]_p, [g]_p \in \mathcal{E}_p$ and $a \in \mathbb{R}$ be given then $[f]_p, [g]_p$ are equivalence classes of maps $f: U_1 \to \mathbb{R}$, $g: U_2 \to \mathbb{R}$ s.t. $p \in U_1$, $p \in U_2$. Put

$[f]_p + [g]_p := [f_{|V} + g_{|V}]_p$ with $V = U_1 \cap U_2$

$[f]_p \cdot [g]_p := [f_{|V} \cdot g_{|V}]_p$ and $a \cdot [f]_p :=$

$[a \cdot f]_p$. Let $1_{\mathcal{E}_p} := [1]_p$ where $1 : U \to \mathbb{R}, x \mapsto 1$

then $\mathbb{R} \to \mathcal{E}_p$, $a \mapsto a \cdot [1]_p$ is a ring

homomorphism, so by def. $\mathcal{E}_p$ is $\mathbb{R}$-alg.

m_p is ideal: $[f]_p, [g]_p \in m_p \iff f(p), g(p) = 0$

$$\Rightarrow \underbrace{f(p) + g(p)}_{=\,((f+g)_{|V})(p)} = 0$$

$$((f+g)_{|V})(p) = 0 \iff [f]_p + [g]_p \in m_p$$

similarly: $[f]_p \in m_p$, $[g]_p \in \mathcal{E}_p$

$$\iff f(p) = 0 \iff ((g \cdot f)_{|V})(p) = 0$$

$$\Updownarrow$$

$$[f]_p \cdot [g]_p \in m_p$$

<u>m_p is a maximal ideal:</u> Suppose that

$\boxed{m_p \subsetneq I \subseteq \mathcal{E}_p} \iff \exists\ \underline{[f]_p \in I}, \ [f]_p \notin m_p$

$\iff \underline{f(p) \neq 0} \iff \exists$ representative $f: U \to \mathbb{R}$ of

$[f]_p$ s.t. $f(p) \neq 0 \iff \exists\ p \in V \subset U : f(x) \neq 0$

$\forall x \in V. \implies$ we can invert f on $V \iff$

$\exists\ [\frac{1}{f}]_p \in \mathcal{E}_p \implies [f]_p \in \mathcal{E}_p^* := \{$units

in $\mathcal{E}_p\} = \{ [f]_p \mid \exists\ [g]_p : [f]_p \cdot [g]_p = [1]_p \}$

$$\xRightarrow[{[f]_p \in \mathcal{E}_p^*}]{[f]_p \in I} [1]_p \in I \iff \boxed{I = \mathcal{E}_p}$$

$\implies m_p$ is a maximal ideal of $\mathcal{E}_p$

$\square$

<u>Rk:</u> $\mathcal{E}_p / m_p$ is a field

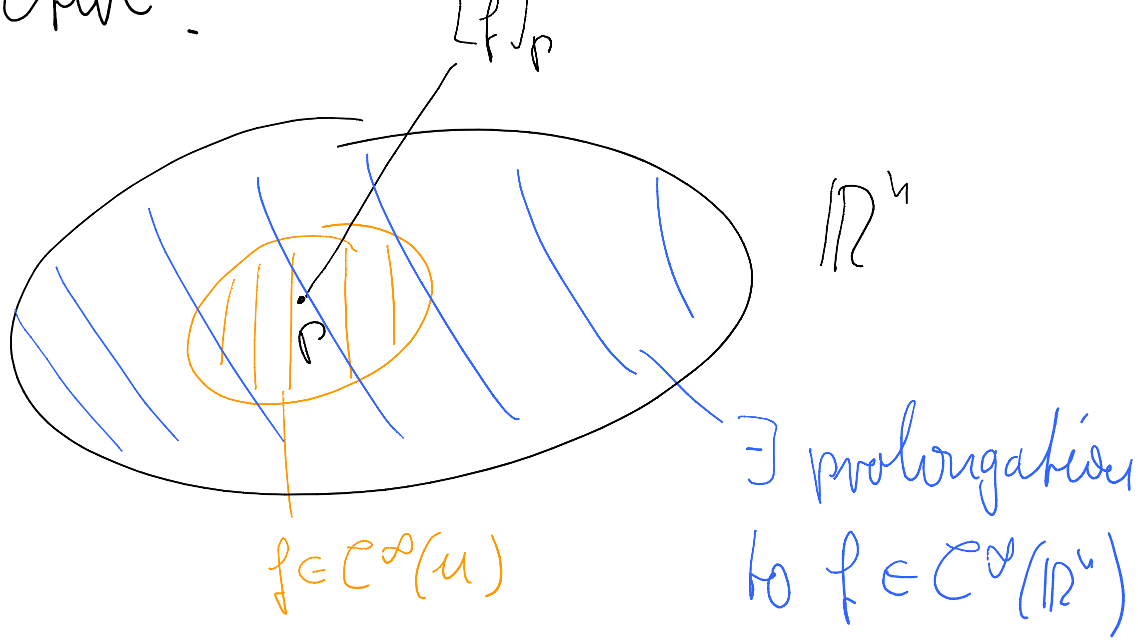
<u>ex:</u> What field?

Lemma 2.6: Let $p \in \mathbb{R}^n$, then the map

$$\phi : \mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathcal{E}_p := \left\{ [f]_p \,\middle|\, f \in \mathcal{C}^\infty(U) \right\}_{p \in U}$$

$$f \longmapsto [f]_p$$

is surjective.

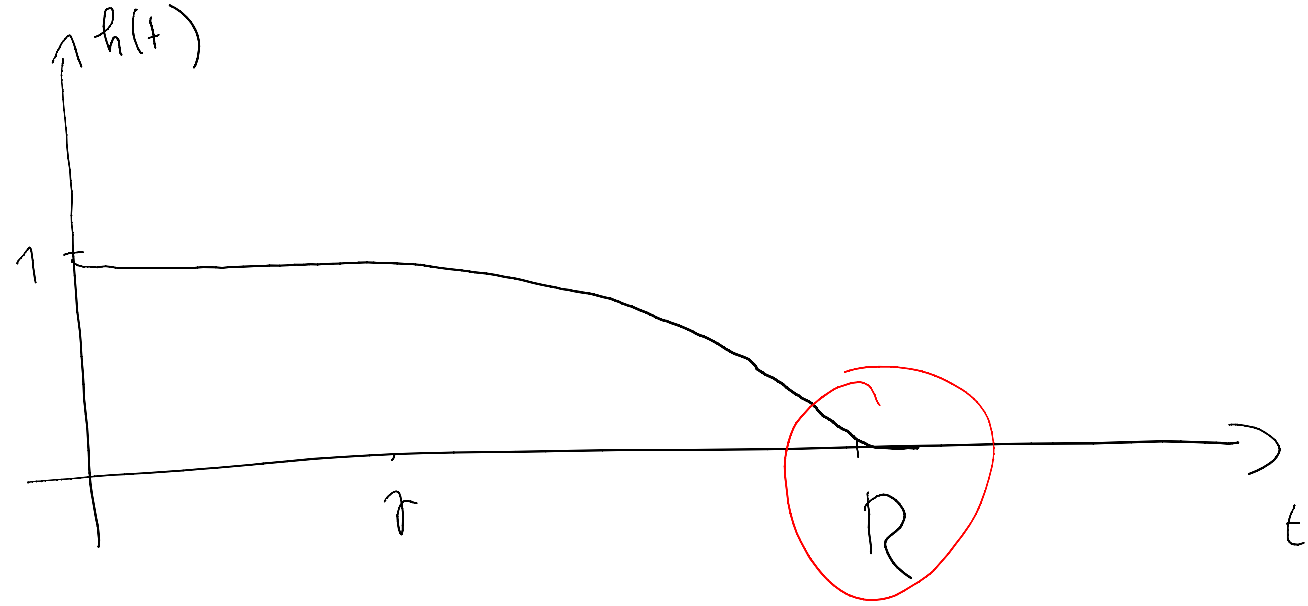


Proof: $\forall\, 0 < r < R < \infty$, define $h : \mathbb{R} \to \mathbb{R}$

$$h(t) := \begin{cases} 0 & \forall\, t \geq R \\ \left(1 + e^{\frac{1}{r-t} + \frac{1}{R-t}}\right)^{-1} & \forall\, t \in (r, R) \\ 1 & \forall\, t \leq r \end{cases}$$

$$\shortparallel$$

$$h_{r,R}(t)$$



<u>Claim</u>: h is smooth (i.e. $h \in C^\infty(\mathbb{R})$)

Pf: like in example in §0.

to show $\lim_{t \downarrow r} h(t) = 1$

$$\lim_{t \downarrow r} h^{(k)}(t) = 0 \quad \forall k > 0$$

$$\lim_{t \nearrow R} h^{(k)}(t) = 0$$

back to the proof: let $[f]_p \in \mathcal{E}_p$ be given, then $\exists\, p \in U \subset \mathbb{R}^n$ open ngh. of p and a representative $f \in C^\infty(U)$ of $[f]_p$. Choose R s.t. $B_p(R) := \{x \in \mathbb{R}^n \mid |x-p| < R\} \overset{!}{\subset} U$

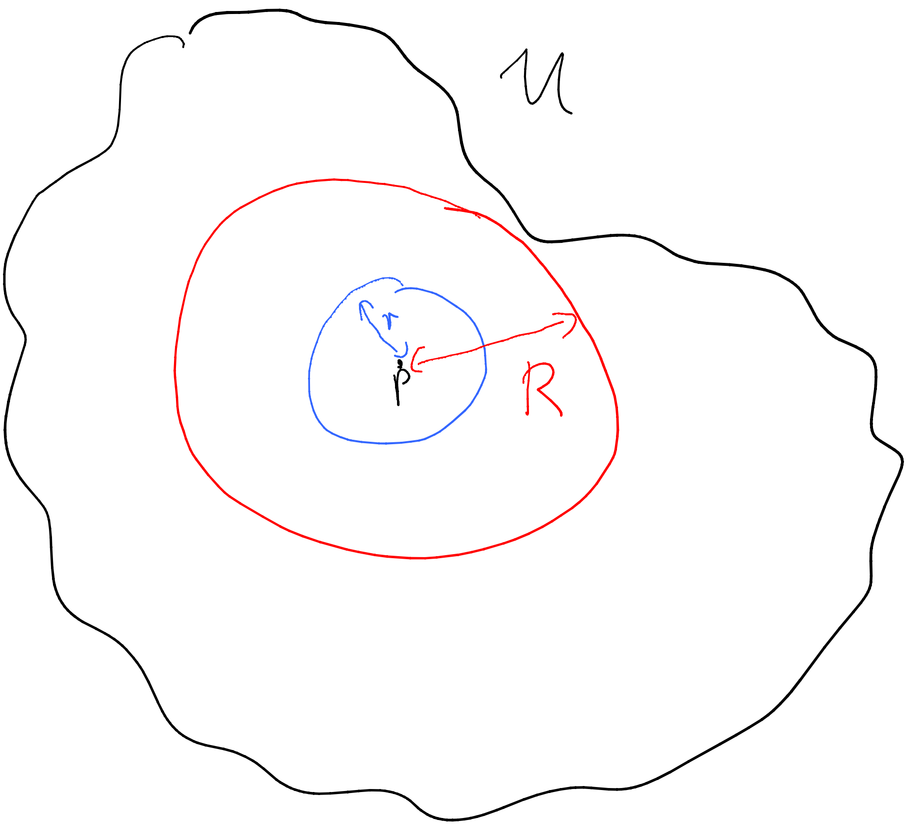
Define $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$g(x) = \begin{cases} f(x) \cdot h(|x-p|) & \forall x \in U \\ 0 & \forall x \in U \end{cases}$$

$h(|x-p|) = h_{R,r}(|x-p|)$ for some $0<r<R$

where $|x| := \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$ is euclidian norm.

obvious: $\forall x \in B_r(p)$: $\underline{g(x) = f(x)}$ (because $h(|x-p|) = 1$)



since $h \in C^\infty(\mathbb{R}) \Longrightarrow g \in C^\infty(\mathbb{R}^n)$. and $[g]_p = [f]_p$ $\square$

From now on, we will develop theory also for (germs of) holomorphic functions a few basic facts from complex analysis:

Let $U \subset \mathbb{C}^n$ open, $f: U \to \mathbb{C} = \mathbb{R}^2$ be smooth (i.e. $f \in \mathcal{C}^\infty(U)$, where $U \subset \mathbb{R}^{2n}$)

<u>Def. 2.7.</u>: a) f is said to be complex differentiable at $a \in U$ if $\forall h \in \mathbb{C}^n$ $\|h\|$ small: $f(a+h) = f(a) + A \cdot h + r$ with $A \in M(1 \times n, \mathbb{C})$, $r \in \mathbb{C}$ s.t. $\lim\limits_{\|h\| \to 0} \frac{|r|}{\|h\|} = 0$. Then $A = Df(a)$

$= (\partial_{z_1} f, \dots, \partial_{z_n} f)(a)$ is the complex derivative of f in a

b) f is called holomorphic on $U \subset \mathbb{C}^n$ if f is complex differentiable $\forall a \in U$.

Write $\mathcal{O}(U) := \{ f : U \to \mathbb{C} \mid f$ holomorphic on $U \}$

have restriction morphisms : $\rho^U_V : \mathcal{O}(U) \to \mathcal{O}(V)$

$\forall V \subset U$ and germs:

$$\mathcal{O}_{\mathbb{C}^n,p} := \mathcal{O}_p := \{ [f]_p \mid f \in \mathcal{O}(U) \text{ for some } p \in U \subset \mathbb{C}^n \}$$

with maximal ideal $m_p := \{ [f]_p \mid f(p) = 0 \} \subset \mathcal{O}_p$

(notation: $\mathcal{O}_n := \mathcal{O}_{\mathbb{C}^n,0}$, $\mathcal{E}_n := \mathcal{E}_{\mathbb{R}^n,0}$)

<u>Theorem 2.8: (Cauchy's integral formula)</u>

Let $U \subset \mathbb{C}^n$, $f \in \mathcal{O}(U)$, $a = (a_1 \dots a_n) \in U$, $\rho_i \in \mathbb{R}_{>0}$ $\forall i \in \{1 \dots n\}$, then $P(a,\rho) := \{ z \in \mathbb{C}^n \mid |z_i - a_i| < \rho_i \}$ is called a polycylinder with multiradius $\rho = (\rho_1, \dots \rho_n)$. Then

$$f(z) = \frac{1}{(2\pi i)^n} \int\limits_{|\zeta_1 - z_1| = \varrho_1} \cdots \int\limits_{|\zeta_n - z_n| = \varrho_n} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1) \cdot \ldots \cdot (\zeta_n - z_n)} \, d\zeta_1 \ldots d\zeta_n$$

<u>Consequence:</u> develop $\frac{f(\zeta_1, \ldots \zeta_n)}{(\zeta_1 - z_1) \cdot \ldots \cdot (\zeta_n - z_n)}$ in z_i

exchange sum and integral $\Rightarrow$

f can be developed in small ngh. of z

<u>in a convergent</u> power series.

study power series development of holomorphic & smooth functions more in detail.

<u>Def 2.9:</u> Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and

$$\mathbb{K}[[x_1, \ldots, x_n]] = \left\{ \sum_{\nu \in \mathbb{N}^n} a_\nu \cdot x^\nu \;\middle|\; a_\nu \in \mathbb{K} \right\}.$$

be the set of formal power series in n variables with coefficients in $\mathbb{K}$.

<u>Lemma 2.10.:</u> $\mathbb{K}[[x_1, \dots, x_n]]$ is $\mathbb{K}$-algebra and $m := \left\{ \sum_{\nu \in \mathbb{N}^n \setminus \{0\}} a_\nu \cdot x^\nu \;\middle|\; a_\nu \in \mathbb{K} \right\} \subset \mathbb{K}[[x_1 \dots x_n]]$ is a maximal ideal.

<u>Pf.:</u> exercise, recall that $+$ and $\cdot$ are defined as:

$$\sum_{\nu \in \mathbb{N}^n} a_\nu \cdot x^\nu + \sum_{\nu \in \mathbb{N}^n} b_\nu \cdot x^\nu = \sum_{\nu \in \mathbb{N}^n} (a_\nu + b_\nu) x^\nu$$

$$\left(\sum_{\nu \in \mathbb{N}^n} a_\nu \cdot x^\nu\right) \cdot \left(\sum_{\nu \in \mathbb{N}^n} b_\nu \cdot x^\nu\right) = \sum_{\nu \in \mathbb{N}^n} \left(\sum_{\lambda + \kappa = \nu} a_\lambda \cdot b_\kappa\right) x^\nu$$

$$c \cdot \sum_{\nu \in \mathbb{N}^n} a_\nu x^\nu = \sum_{\nu \in \mathbb{N}^n} (c \cdot a_\nu) x^\nu$$

$\square$

Statement about power series / Taylor development of (germs of) smooth / hol. functions:

<u>Definition + Theorem 2.11</u>: Let $\mathcal{R}_n := \begin{cases} \mathcal{E}_{\mathbb{R}^n,0} \\ \mathcal{O}_{\mathbb{C}^n,0} \end{cases}$

(i.e. $\mathcal{R}_n$ is $\mathbb{K}$-algebra), the the map

$$T: \mathcal{R}_n \longrightarrow \mathbb{K}[[x_1, \dots, x_n]]$$

$$f \longmapsto \sum_{\nu \in \mathbb{N}^n} a_\nu x^\nu \quad \text{with}$$

$$a_\nu := \frac{1}{\nu!} \left(\frac{\partial^\nu}{\partial x^\nu} f \right)(0)$$

is called the Taylor development of f at 0. T is a $\mathbb{K}$-algebra homomorphism and $T: \mathcal{O}_n \to \mathbb{C}[[x_1 \dots x_n]]$ is injective,

not surjective

$T: \mathcal{E}_n \to \mathbb{R}[[x_1 \dots x_n]]$ is $\boxed{\text{surjective}}$

not injective.

<u>Easy parts of the proof:</u>

1.) well-defined: let $[f]_0 = [g]_0$ be given

then $\exists\, U \in \mathbb{K}^n$ open, $0 \in U$: $f_{|U} = g_{|U}$

$$\Rightarrow (D^\nu f)_{|U} = (D^\nu g)_{|U} \Rightarrow (D^\nu f)(0) = (D^\nu g)(0)$$

$$\Rightarrow T([f]_0) = T([g]_0)$$

2.) T is homomorphism:

$$T([f]+[g]) = T([f+g]) = \sum \frac{D^\nu(f+g)(0)}{\nu!} x^\nu$$

$$= \sum \frac{(D^\nu f)(0) + (D^\nu g)(0)}{\nu!} x^\nu = T([f]) + T([g])$$

$$c \cdot T([f]) = T([c \cdot f]) = T(c \cdot [f])$$

$\boxed{\text{ex}}$: $T([f] \cdot [g]) = T([f]) \cdot T([g])$

use Leibniz rule:

$$D^\nu (f \cdot g) = \sum_{\kappa + \lambda = \nu} \binom{\nu}{\lambda} D^\kappa f \cdot D^\lambda g$$

with $\binom{\underline{\nu}}{\underline{\lambda}} = \dfrac{\underline{\nu}!}{\underline{\lambda}! \cdot (\underline{\nu} - \underline{\lambda})!}$

3.) $T : \mathcal{O}_n \to \mathbb{C}[[\underline{x}]] := \mathbb{C}[[x_1, \dots, x_n]]$ injective:

Taylor series of hol. function is convergent in small ngh. of 0, i.e. it defines a hol. function there, namely f. Hence we can get f back from $T(f)$. T not surj.!

e.g. $\sum n!\, x^n$ is divergent and does not appear as Taylor series of some $f \in \mathcal{O}_n$.

4.) $T : \mathcal{E}_n \to \mathbb{R}[[x]]$ not injective:

$\exists\, [f] \in \mathcal{E}_n$ with $[f] \neq [0]$ but $(D^\nu f)(0) = 0$ $\forall \nu \in \mathbb{N}$ (e.g. $\S 0$ or $h : \mathbb{R} \to \mathbb{R}$ above)

<u>remains</u>: <u>T is surjective</u> !