Lemma 1.4.: $U \subset \mathbb{R}^n$ star-shaped, center $p \in U$,

$f: U \to \mathbb{R}$ smooth. $k \in \mathbb{N}_{>0}$ s.t. $(D^\ell f)(p) = 0 \ \forall \ell \in \{0, \dots, k-1\}$

Then $\forall i_1 \dots i_k \in \{1, \dots, n\}: \exists g_{i_1 \dots i_k}$ smooth s.t.

$$\boxed{f(x) = \sum_{i_1 \dots i_k = 1}^{n} g_{i_1 \dots i_k}(x) \cdot (x_{i_1} - p_{i_1}) \cdot \ldots \cdot (x_{i_k} - p_{i_k})}$$

& $g_{i_1 \dots i_k}(p) = \frac{1}{k!} (\partial_{x_{i_1}} \dots \partial_{x_{i_k}} f)(p)$

Lemma 1.6: Let $0 \in U \subset \mathbb{R}$ open and $f: U \to \mathbb{R}$ smooth, then $\exists\ 0 \in V \subset U$ and local diffeom. $\varphi: V \to V$, $\varphi(0) = 0$ s.t. $f(\varphi(y)) = \varepsilon^{k-1} \cdot y^k$ for some $k \in \mathbb{N}$, $\varepsilon \in \{1, -1\}$ iff $k = \min\{\ell \in \mathbb{N} \mid f^{(\ell)}(0) \neq 0\}$. If k is even $\Rightarrow \varepsilon = f^{(k)}(0) / |f^{(k)}(0)|$

Proof: Let $k = \min\{\ell \in \mathbb{N} \mid f^{(\ell)}(0) \neq 0\}$ and put $\varepsilon = f^{(k)}(0) / |f^{(k)}(0)|$. Using 1.4:

$\exists\ g$ smooth, defined on a star-shaped set

V with center 0, i.e. $V = (a,b) \ni 0$ s.t.

$f(x) = x^k \cdot g(x)$ $\forall x \in V$. and $g(0) = \frac{1}{k!} f^{(k)}(0) \neq 0$

Hence: $\underline{\varepsilon \cdot g(0)} > 0 \Rightarrow \varphi(x) := \varepsilon \cdot x \cdot \sqrt[k]{\varepsilon \cdot g(x)}$

is smooth on $V' \subset V$ for $0 \in V'$ small &

$\varphi(0) = 0$ and $\varepsilon^{k-1} \cdot \varphi(x)^k = \varepsilon^{k-1} \cdot \left(\varepsilon x \sqrt[k]{\varepsilon g(x)}\right)^k = \varepsilon^{2k} x^k g(x)$

$= x^k g(x) = f(x)$. Moreover, $\varphi'(0) = \varepsilon \sqrt[k]{\varepsilon g(0)} \neq 0$

This implies that φ is local diffeomorphism

at $0 \in V' \Rightarrow \exists \overset{0}{\underset{\cap}{}} V'' \subset V'$ and an inverse map

$\psi : V'' \to V''$: $\psi(0) = 0$ and $f(\psi(y)) = \varepsilon^{k-1} \cdot y^k$ $\square$

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other simple example for classification.

Lemma 1.7: Let $U \subset \mathbb{R}^n$ star-shaped, center $0 \in U$.

$f : U \to \mathbb{R}$ smooth, $f(0) = 0$. Then

$$\left.\begin{array}{l} 0 \text{ is non-critical} \\ \text{pt. of } f \text{, i.e.} \\ (Df)(0) \neq 0 \end{array}\right\} \iff \left\{\begin{array}{l} \exists \text{ local diffeomorphism} \\ \psi \text{ at } 0,\ \psi(0) = 0 \text{ s.t.} \\ f(\psi(y)) = y_1 \ \forall (y_1 \dots y_n) \in V \subset U \end{array}\right.$$

Proof: "$\Leftarrow$": If $\exists\, \varphi$ with $f(\varphi(y)) = y_1$

$$\Rightarrow D\big(f(\varphi(y))\big) = (1,0\dots 0) \neq (0,\dots 0)$$

$\Rightarrow$ 0 is not critical for $f(\varphi(y))$

$\underset{\text{last time}}{\Rightarrow}$ 0 is not critical for f.

"$\Rightarrow$": Suppose now: $(Df)(0) \neq (0,\dots,0) \in \mathbb{R}^n$, i.e.

$\exists\, i \in \{1,\dots,n\}: (\partial_{x_i} f)(0) \neq 0 \in \mathbb{R}$. Supp. $i \neq 1$,

define $\varphi: U \to U: (y_1,\dots,y_n) = (\varphi_1(x),\dots,\varphi_n(x)) =$

$$(f(x), x_2,\dots,x_{i-1}, x_1, x_{i+1},\dots,x_n)$$

$$(D\varphi) = \begin{bmatrix} \partial_{x_1} f & 0 & & 0 & 1 & & 0 \\ \partial_{x_2} f & 1 & \ddots & 0 & \vdots & & \\ \vdots & 0 & & 1 & \vdots & & \\ \vdots & \vdots & & 0 & 0 & & \\ & \vdots & & \vdots & \vdots & 1 & \ddots \\ \partial_{x_n} f & 0 & & 0 & 0 & 0 & 1 \end{bmatrix} : U \to M(n \times n, \mathbb{R})$$

$$\det(D\varphi(0)) =$$

$$-\det \begin{bmatrix} \partial_{x_i} f & & & 0 \\ \partial_{x_2} f & 1 & & \\ \vdots & & \ddots & \\ \partial_{x_1} f & & & \\ \vdots & 0 & & 1 \\ \partial_{x_n} f & & & \end{bmatrix} (0) = (-\partial_{x_i} f)(0) \neq 0$$

$\Rightarrow \det(D\phi(0)) \neq 0$, hence φ is local diffeom.

Let $\psi : V \to V$ be the inverse map, defined on $0 \in V \subset U \Rightarrow f(\psi(y)) = y_1$. If $i = 1$, then exchange x_1 with some x_i $(i \neq 1)$, this is a global diffeomorphism of $\mathbb{R}^n$. $\square$

Theorem 1.8. (Morse lemma): Let $f : U \to \mathbb{R}$ smooth $0 \in U$, $f(0) = 0$. Then 0 is a non-degenerate critical point of f iff $\exists$ local diffeomorphism ψ at 0, $\psi(0) = 0$ s.t. $f(\psi(y)) = y_1^2 + \dots + y_k^2 - y_{k+1}^2 - \dots - y_n^2$ here $n - k =: l$ is index $(D^2 f)(0)$.

Proof: "$\Leftarrow$": $D^2(f(\psi(0)))$ is

$$\left.\begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{bmatrix}\begin{matrix} \left.\vphantom{\begin{matrix}1\\ \ddots\\ 1\end{matrix}}\right\} k \\ \left.\vphantom{\begin{matrix}-1\\ \ddots\\ -1\end{matrix}}\right\} \ell \end{matrix}\right\} \Rightarrow \operatorname{rk} D^2 f(\psi(0)) = n$$

$\Rightarrow \operatorname{rk}(D^2 f)(0) = n \Rightarrow 0$ is non-deg. pt. of f.

"$\Rightarrow$": Lemma 1.4 $\Rightarrow \exists$ smooth functions

g_{ij} $(\forall i,j \in \{1 \dots n\})$: $\boxed{f(x) \overset{(*)}{=} \sum_{i,j=1}^{n} x_i x_j \cdot g_{ij}(x) \ \forall x \in U.}$

& $g_{ij}(x) = g_{ji}(x)$. Put $G(x) := (g_{ij}(x))_{i,j=1\dots n}$:

$U \to \underbrace{Sym(n, \mathbb{R})}$ Space of symmetric matrices of size n

$G(0) = (D^2 f)(0)$ and

$$\boxed{f(x) = x^{tr}\cdot G(x)\cdot x\,,\quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} \iff (*)$$

Aim: Find local diffeomorphism at $0 \in U$ which transforms $G(x)$ in constant diagonal matrix with $1, -1$ as diagonal entries.

obvious: $\exists\, S \in Gl(n,\mathbb{R}) : S^{tr}\cdot G(0)\cdot S = D$

$:= \operatorname{diag}(\lambda_1 \dots \lambda_k, \varepsilon_1 \dots, \varepsilon_l)$ $\lambda_i = 1$, $\varepsilon_j = -1$, $k+l=n$

$l = \operatorname{index}(G(0))$. Consider space $\mathcal{T}(n,\mathbb{R})$ of upper triangular matrices, and the map

$$\alpha : \mathcal{T}(n,\mathbb{R}) \longrightarrow \operatorname{Sym}(n,\mathbb{R})$$

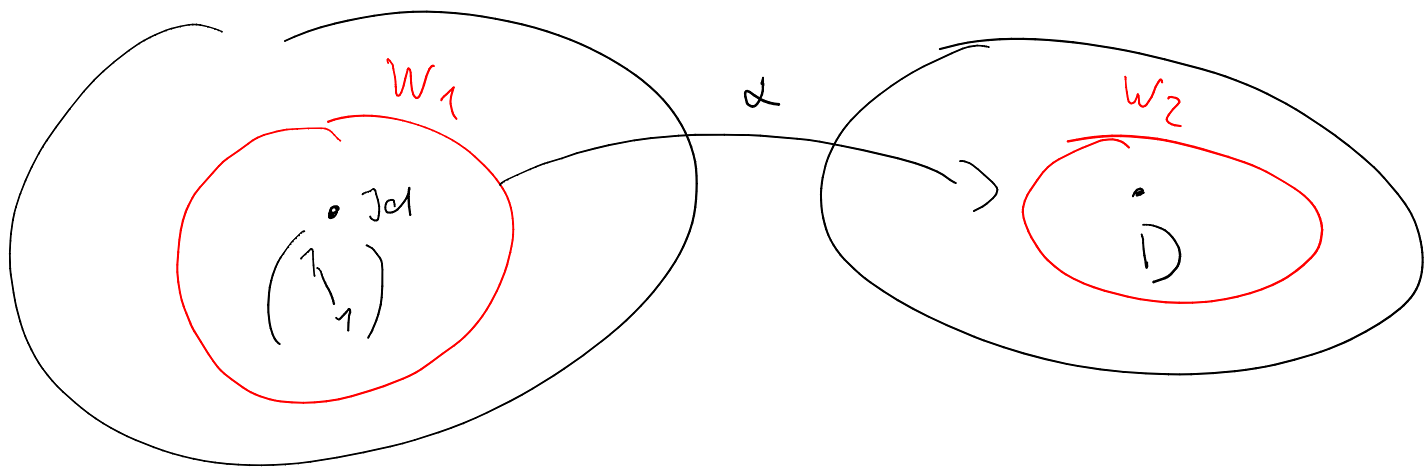
$$T \longmapsto (S\cdot T)^{tr}\cdot G(0)\cdot(S\cdot T)$$

$$= T^{tr}\underbrace{\left(S^{tr}\cdot G(0)\cdot S\right)}_{D}\cdot T = T^{tr}\cdot D\cdot T$$

<u>Clear</u>: - $\mathrm{Id} \in \mathcal{T}(n,\mathbb{R})$, $\alpha(\mathrm{Id}) = S^{tr}\, G(0) \cdot S = D$

- both $\mathcal{T}(n,\mathbb{R})$ and $\mathrm{Sym}(n,\mathbb{R})$ are $\mathbb{R}$-vector spaces of dim $\frac{1}{2}n(n+1)$.

<u>Hence</u>: α is a smooth map, which we consider as defined from $W_1 \subset \mathcal{T}(n,\mathbb{R})$ to $W_2 \subset \mathrm{Sym}(n,\mathbb{R})$, where W_1 resp. W_2 are small ngh. of $\mathrm{Id} \in \mathcal{T}(n,\mathbb{R})$ resp. $D \in \mathrm{Sym}(n,\mathbb{R})$.



$$\mathcal{T}(n,\mathbb{R}) \;\widehat{=}\; \mathbb{R}^{\frac{1}{2}n(n+1)} \;\widehat{=}\; \mathrm{Sym}(n,\mathbb{R})$$

Then $(D\alpha)(\mathrm{Id})$ is a linear map between

$\mathcal{T}(n,\mathbb{R})$ and $\mathrm{Sym}(n,\mathbb{R})$ (analysis 2).

namely: $(D\alpha)(\mathrm{Id})(T) = (S\cdot T)^{tr}\cdot G(0) + G(0)\cdot S T$.

<u>Claim</u>: it is injective (use $G(0)\neq 0$)

hence also surjective $\Rightarrow$ an isomorphism,

$\Longrightarrow$ α is local diffeomorphism at $\mathrm{Id}\in\mathcal{T}(n,\mathbb{R})$ with inverse map β.

<u>Hence</u>: $\exists\ 0\in V\subset U$ open: $\forall x\in V$:

$$G(x) = \Big(\underbrace{(\beta\circ(S^{tr}\cdot G\cdot S))(x)}_{T(x)}\Big)^{tr}\cdot D\cdot\big((\beta\circ(S^{tr}\cdot G\cdot S))(x)\big)$$

<u>Define</u>: $\phi: V\to V$, $x\mapsto \underbrace{(\beta\circ(S^{tr}G S))(x)}_{\in\,\mathcal{T}(n,\mathbb{R})}\cdot x =: y$

then ϕ smooth, $\phi(0)=0$ and

$$(D\phi)(0) = \left((\beta \circ (S^{tr} G S))\right)(0) = \beta \circ (S^{tr} \cdot G(0) \cdot S) = \beta(D) = \mathrm{Id}$$

$\Rightarrow \det(D\phi(0)) \neq 0 \Rightarrow \phi$ is local diffeo.

at $0 \in V$ with inverse map $x := \psi(y)$.

recall: $f(x) = x^{tr} \cdot G(x) \cdot x =$

$$x^{tr} \cdot \left[(\beta \circ (S^{tr} G S))(x) \cdot D \cdot (\beta \circ (S^{tr} G S))(x) \right] \cdot x$$

$$= \underbrace{\left(\beta \circ (S^{tr} G S)(x) \cdot x \right)}_{\phi(x)}^{tr} \cdot D \cdot \underbrace{\left((\beta \circ (S^{tr} G S))(x) \cdot x \right)}_{\phi(x)}$$

$$= \phi(x)^{tr} \cdot D \cdot \phi(x)$$

$$\Rightarrow f(\psi(y)) = \phi(\psi(y))^{tr} \cdot D \cdot \phi(\psi(y))$$

$\phi \circ \psi = \mathrm{Id}_V$ $\quad = y^{tr} \cdot D \cdot y$

$$(f \circ \psi)(y) = y^{tr} \cdot D \cdot y = y_1^2 + \dots + y_k^2 - y_{k+1}^2 - \dots - y_n^2$$

$\square$

§2. Map germs, Taylor development, local rings and the lemma of Nakayama.

Aim: precise framework for local investigations.

need: elementary notions from algebra like rings, ideals, modules ....

Def. 2.1: Let R be ring. A ring S is called R-algebra, if $\exists$ ring homomorphism $R \to S$, in other words: An R-module S together with a multiplication $S \times S \to S$, compatible with scalar mult. $R \times S \to S$ is called an R-algebra.

<u>Def.-Lemma 2.2.:</u> Let $U \subset \mathbb{R}^n$ open,

put $C^\infty(U) := \{ f: U \to \mathbb{R} \mid f \text{ smooth} \}$

The $C^\infty(U)$ with point-wise addition + multiplication is an $\mathbb{R}$-algebra.

$\forall V \subset U$ open : $\exists$ restriction maps :

$$\begin{aligned} \rho^U_V : C^\infty(U) &\to C^\infty(V) \\ f &\longmapsto f_{|V} \end{aligned} \quad \begin{pmatrix} \mathbb{R}\text{-algebra-} \\ \text{homomorphism} \end{pmatrix}$$

& $\forall W \subset V \subset U : \rho^U_W = \rho^V_W \circ \rho^U_V$

<u>Def. 2.3:</u> (germs of sets and maps)

i) Let $A, B \subset \mathbb{R}^n$ (more generally, $A, B \subset X$ where X is a topological space), $p \in \mathbb{R}^n$

We say $A \overset{p}{\sim} B$ iff:

$\exists$ open neighbh. U of p in $\mathbb{R}^n$:

$A \cap U = B \cap U$.

This is an equivalence rel. on $\mathcal{P}(\mathbb{R}^n) = \{A \subset \mathbb{R}^n\}$, write (A,p) for the equivalence class of A w.r.t. $\overset{p}{\sim}$. (A,p) is called the germ of the set A at p

ii) let $Y \subset \mathbb{R}^m$ be open, let $f: A \to Y$, $g: B \to Y$ maps to Y, let $p \in \mathbb{R}^n$.

Define $f \overset{p}{\sim} g \iff \exists\, U$ open neighb. of p in $\mathbb{R}^n$: $f_{|U \cap A} = g_{|U \cap B}$. The equivalence class of f w.r.t. $\overset{p}{\sim}$ is called (map) germ of f at p, written (f,p) or $[f]_p$.