recall: $f \in m^2_{R_n}$, corank$(f) = 2$, $\mu(f) \leq 5$

$\Rightarrow$ f stably equivalent to $g \in m^3_{R_2}$, $\mu(g) = \mu(f) \in \{4, 5\}$

write $g = g_3 + g_4 + \dots$ $g_k \in \mathbb{K}[x,y]_k$, then

Theorem 4.8: $\exists$ linear coord. change ϕ_A s.t.

$\phi_A^* g_3 \in \{0, x^3, x^2y, x^3 - xy^2, x^3 + y^3\}$

equi. eq. for $\mathbb{K} = \mathbb{C}$

Theorem 4.10: $f \in m^2_{R_n}$, Corank$(f) = 2$, $\mu(f) = 4$

$\Rightarrow$ f is stably equivalent to either

$x^3 - xy^2$ or $x^3 + y^3$ (both equivalent for $R_n = \mathcal{O}_n$)

Proof: Splitting lemma (Th. 4.2): f is stably equivalent to $g(x,y) \in m^3_{R_2}$ with $\mu(g) = 4$. Show: determinacy of g is 3:

$g \in m^3_{R_2} \Rightarrow J_g \subset m^2_{R_2} \Rightarrow m_{R_2} \cdot J_g \subset m^3_{R_2}$

We have:

$$\boxed{\dim_{\mathbb{K}} \frac{R_2}{m_{R_2}\cdot J_g} \overset{(*)}{=} \dim_{\mathbb{K}} \frac{R_2}{m_{R_2}^3} + \dim_{\mathbb{K}} \frac{m_{R_2}^3}{m_{R_2}^3\cdot J_g}}$$

we have $R_2/m_{R_2}^3 = \bigoplus_{i+j<3} \mathbb{K}\cdot x^i y^j$

so $\dim_{\mathbb{K}} R_2/m_{R_2}^3 = 6$. Moreover, by

Lemma 4.6 : $J_g = m_{R_2}\cdot J_g \oplus \mathbb{K}\partial_x g \oplus \mathbb{K}\partial_y g$

hence: $\dim_{\mathbb{K}} R_2/m_{R_2}\cdot J_g = \underbrace{\dim R_2/J_g}_{\substack{\shortparallel \\ \mu(g)}} + 2$

$\mu(g) + 2 = 4+2 = 6$

hence $\dim {}^{R_2}/_{m_{R_2} J_g} = \dim {}^{R_2}/_{m_{R_2}^3}$

$\overset{(*)}{\Longrightarrow} \dim {}^{m_{R_2}^3}/_{m_{R_2} J_g} = 0$

meaning: $m_{R_2}^3 = m_{R_2} \cdot J_g$

$\Longrightarrow m_{R_2}^4 \subset m_{R_2}^2 J_g \overset{3.10}{\Longrightarrow} g$ is 3-det.

i.e. $g \underset{R}{\sim} T_g^3$. Since $g \in m_{R_2}$,

we have $T_g^3 = g_3 \in \mathbb{K}[x,y]_3$

$\overset{4.8., 4.9}{\Longrightarrow} T_g^3 \sim \begin{cases} x^3+y^3 \\ x^3-xy^2 \end{cases}$

$\square$

Next case: $\mu(f)=\mu(g)=5$.

Theorem 4.11.: Let $f \in m_{\mathcal{R}_n}^2$, $\operatorname{corank}(f)=2$, $\mu(f)=5$. Then f is stably equivalent to x^2y+y^4 or $-x^2y-y^4$ (both are equivalent for $\mathcal{R}_n = \mathcal{O}_n$).

Pf: by splitting lemma, f stably equivalent to $g \in m_{\mathcal{R}_2}^3$, $\mu(g)=5$. Show: determinacy of $g = 4$. Again, we have

$$g \in m_{\mathcal{R}_2}^3 \Rightarrow J_g \subset m_{\mathcal{R}_2}^2 \Rightarrow m_{\mathcal{R}_2} \cdot J_g \subset m_{\mathcal{R}_2}^3$$

and $\dim_{\mathbb{K}} \mathcal{R}_2 / m_{\mathcal{R}_2} \cdot J_g = \underline{\dim \mathcal{R}_2 / m_{\mathcal{R}_2}^3} + \dim m_{\mathcal{R}_2}^3 / m_{\mathcal{R}_2} \cdot J_g$

$\underline{4.6.} \Longrightarrow \dim_{\mathbb{K}} R_2/\mathfrak{m}\cdot J_g = \underbrace{\dim R_2/J_g}_{\mu(g)=5} + 2 = 7$

$$\Rightarrow \dim \frac{\mathfrak{m}_{R_2}^3}{\mathfrak{m}_{R_2}\cdot J_g} = 7 - \dim R_2/\mathfrak{m}_{R_2}^3 = 7-6 = 1$$

(i.e. $\mathfrak{m}_{R_2} J_g \subsetneq \mathfrak{m}_{R_2}^3$). As in proof of 3.5:

$$\mathfrak{m}_{R_2}^3/\mathfrak{m}_{R_2} J_g = \frac{\mathfrak{m}_{R_2}^3 + \mathfrak{m}_{R_2} J_g}{\mathfrak{m}_{R_2} J_g} \supsetneq \frac{\mathfrak{m}_{R_2}^4 + \mathfrak{m}_{R_2} J_g}{\mathfrak{m}_{R_2} J_g}$$

$$\Rightarrow \dim_{\mathbb{K}} \frac{\mathfrak{m}_{R_2}^4 + \mathfrak{m}_{R_2} J_g}{\mathfrak{m}_{R_2} J_g} < \dim_{\mathbb{K}} \frac{\mathfrak{m}_{R_2}^3}{\mathfrak{m}_{R_2}\cdot J_g} = 1$$

$$\Rightarrow \dim_{\mathbb{K}} \frac{\mathfrak{m}_{R_2}^4 + \mathfrak{m}_{R_2}\cdot J_g}{\mathfrak{m}_{R_2}\cdot J_g} = 0$$

hence $m_{R_2}^4 \subset m_{R_2}^4 + m_{R_2} \cdot J_g \overset{!}{\subset} m_{R_2} \cdot J_g$

hence $m_{R_2}^5 \subset m_{R_2}^2 \cdot J_g \overset{3.10}{\Longrightarrow} g$ is 4-det.

Clearly, g is not 3-det, since in this case, we would have $g \sim T_g^3 = g_3$ and then $g_3 \in \{x^3+y^3, x^3-xy^2\}$ $\lightning$ $\mu(g)=4$.

So determinacy $(g)=4$. This means:

$g \underset{R}{\sim} g' := p + h$ with $p \in \mathbb{K}[x,y]_3$ $h \in \mathbb{K}[x,y]_4$. Now $\forall A \in Gl(2,\mathbb{K})$, we have $\phi_A^*(p+h) = p' + h'$ with

$p' \in \mathbb{K}[x,y]_3$, $h' \in \mathbb{K}[x,y]_4$ and $\phi_A(x,y) := A \cdot \binom{x}{y}$. Hence we can assume by Th. 4.8:

$p \in \{0, x^3, x^2y, x^3+y^3, x^3-xy^2\}$. Show: $\underline{p = x^2y}$

<u>Supp.</u>: $p = 0 \Longrightarrow g' = h \in m_{R_2}^4 \Longrightarrow J_{g'} \subset m_{R_2}^3$

$$\mu(f) = \mu(g) = \mu(g') = \dim R_2/J_{g'} \geq \dim R_2/m_{R_2}^3 = 6$$

$\lightning$ $\mu(f) = 5$

<u>Supp</u>: $p = x^3 - xy^2$ or $p = x^3 + y^3$ since

p is 3-det $\Longrightarrow g' = p + h \underset{R}{\sim} p$

$\lightning$ $\mu(f) = \mu(g) = \mu(g') = \mu(p) = 4$ $\lightning$

supp: $p = x^3$,

write $h(x,y) = a \cdot y^4 + b\,x y^3 + x^2 \cdot q(x,y)$

with $q \in \mathbb{k}[x,y]_2$. Put $\varphi(x,y) = \left(x - \frac{1}{3} q(x,y), y\right)$

$$\Rightarrow (D\varphi)(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \varphi \in \mathcal{G}_2 \text{ and}$$

$$\varphi^* g' = \varphi^* \left(\underline{x^3} + x^2 q(x,y) + b x y^3 + a y^4\right)$$

$$= \left(x - \tfrac{1}{3} q\right)^3 + \left(x - \tfrac{1}{3} q\right)^2 \cdot \left(\varphi^* q\right) + b\left(x - \tfrac{1}{3} q\right) \cdot y^4 + a y^4$$

$$= x^3 - x^2 \cdot q + \underbrace{\ldots\ldots}_{\in \mathfrak{m}^5_{R_2}} + x^2 \cdot \varphi^* q - \underbrace{\ldots\ldots}_{\in \mathfrak{m}^5_{R_2}} + b x y^4 - \underbrace{\ldots\ldots}_{\in \mathfrak{m}^5_{R_2}} + a y^4$$

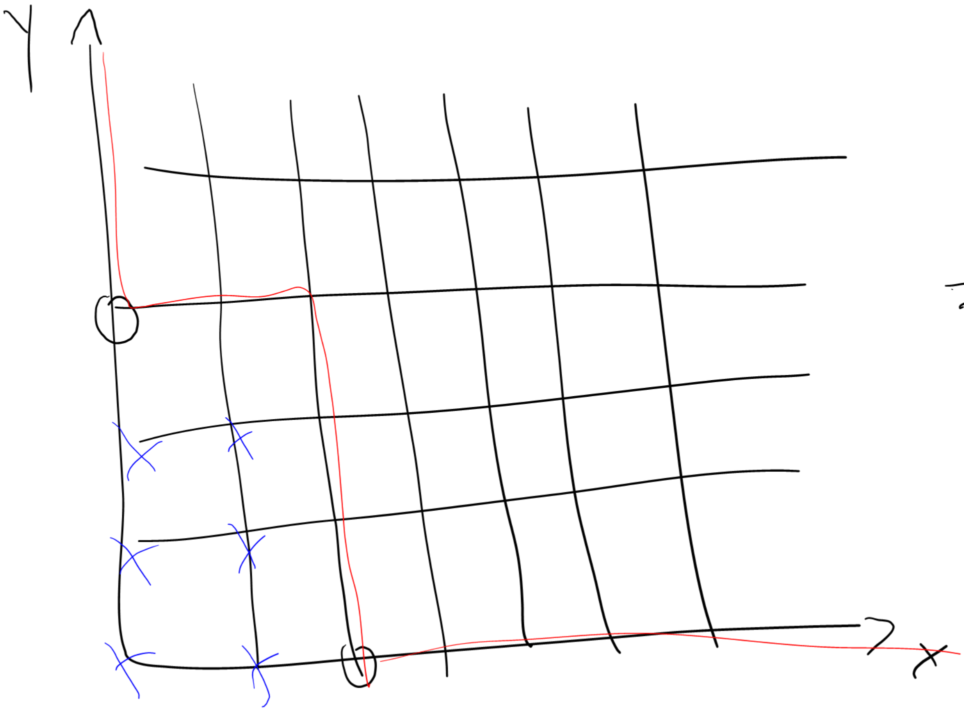
$$= x^3 + b x y^4 + a y^4 - x^2 q + \underbrace{x^2 \varphi^* q}_{= x^2 \cdot q + \ldots\ldots} + \mathfrak{m}^5_{R_2}$$

$$\in x^3 + a y^4 + b x y^3 + \mathfrak{m}^5_{R_2}$$

f, g, g' are 4-det $\Rightarrow g' \sim g'' := x^3 + ay^4 + bxy^3$

<u>Claim</u>: $\mu(g'') > 5$:

let $b = 0 \Rightarrow J_{g''} = (x^2, ay^3)$ and



$\Rightarrow \mu(g'') = 6$

---

let $b \neq 0$, suppose that we have a relation:

$\underline{v + \alpha \bar{x} + \beta \bar{y} + \gamma \bar{x}\bar{y} + \delta \bar{y}^2 + \varepsilon \bar{y}^3} = 0$ in $R_2 / J_{g''}$

$v, \alpha, \beta, \gamma, \delta, \varepsilon \in \mathbb{K}$

i.e.: $\exists$ relation:

$$\underline{m} + \alpha x + \beta y + \gamma xy + \delta y^2 + \varepsilon y^3 + \underline{h_1(x,y)} \cdot \overbrace{\left(3x^2 + \underline{b y^3}\right)}^{\partial_x g^4 \,=}$$

$$+ h_2(x,y) \cdot \left(\underbrace{4ay^3 + 3bxy^2}_{\partial_y g^4}\right) = 0 \text{ in } R_2$$

$$\Rightarrow \underbrace{m}_{\text{coeff of } 1} = 0 \,;\; \underbrace{\alpha}_{\text{coeff of } x} = 0 \,,\; \underbrace{\beta}_{\text{coeff of } y} = 0$$

$$\underbrace{\gamma}_{\text{coeff of } xy} = 0 \,,\; \underbrace{\delta}_{\text{coeff of } y^2} = 0 \,,\; \underbrace{3 \cdot h_1(0,0)}_{\text{coeff of } x^2} = 0$$

$$\underbrace{3b \cdot h_2(0,0)}_{\text{coeff. of } xy^2} = 0 \,,\; \underbrace{\varepsilon + b \cdot h_1(0,0) + 4a h_2(0,0)}_{\text{coeff. of } y3} = 0$$

$h_1(0,0) = h_2(0,0) = 0 \Rightarrow \varepsilon = 0$

$\Rightarrow \alpha = \beta = \gamma = \delta = \varepsilon = 0$

$\Rightarrow \bar{1}, \bar{x}, \bar{y}, \overline{xy}, \bar{y}^2, \bar{y}^3$ are lin. ind.

in $R_2 / J_{g''}$ $\Rightarrow \mu(g'') > 5$

hence $p \neq x^3$

hence $p = x^2 y$

---

we have shown: $\underline{g' = x^2 \cdot y + h(x,y)}$ with

$h(x,y) = a \cdot y^4 + b x y^3 + x^2 \cdot q(x,y)$, $q \in \mathbb{K}[x,y]_2$

Put now $\varphi(x,y) = (x + \frac{b}{2} y^2, y + q) \Rightarrow \varphi \in G_2$

and $(\varphi^{-1})^* g' \in \underline{x^2 \cdot y + a y^4} + m_{R_2}^5$, since

$$\varphi^*\left(x^2y+ay^4\right)=\underline{\left(x+\frac{b}{2}y^2\right)^2}(y+q)+$$

$$\underline{a\cdot(y+q)^4}$$

$$\in\left(x^2+bxy^2\right)(y+q)+ay^4+m_{\mathbb{R}_2}^5$$

$$\in\underbrace{x^2y+bxy^3+qx^2+ay^4}_{=g'}+m_{\mathbb{R}_2}^5$$

g' is 4-det $\Rightarrow g'\sim x^2y+ay^4$

and $a\in\mathbb{K}\setminus\{0\}$:

let $a>0$ (if $\mathbb{K}=\mathbb{R}$)

$\Rightarrow A=\mathrm{diag}\left(a^{1/8},a^{-1/4}\right)$

$a < 0 \Rightarrow A := \operatorname{diag}\left(|a|^{1/8}, -|a|^{1/4}\right)$

$$\Rightarrow \phi_{A}^{*}\left(x^2y + ay^4\right) = \begin{cases} y^4 + x^2y & a > 0 \\ -y^4 - x^2y & a < 0 \end{cases}$$

$\square$

<u>Lemma 4.12.:</u> If $R_n = E_n \Rightarrow y^4 + x^2y \not\sim_R -y^4 - x^2y$

Summarizing:

<u>Theorem 4.13.:</u> Let $f \in m^2_{R_n}$, $1 < \mu(f) \leq 5$.

Then f is stably equivalent to one of the following list, which are pairwise inequivalent for $R_n = E_n$ (for $R_n = O_n$ further equivalences are written)

f	corank(f)	$\mu(f)$	del(f)
x^3	1	2	3
$x^4, -x^4$ eg. $R_n=O_n$	1	3	4
x^5	1	4	5
$x^6, -x^6$ eg. for $R_n=O_n$	1	5	6
x^3-xy^2	2	4	3
x^3+y^3 eg. for $R_n=O_n$	2	4	3
x^2y+y^4	2	5	4
$-x^2y-y^4$	2	5	4

<u>Rmk:</u> One can go on the same way for $\mu > 5$. From some μ on, one has continuous families with constant Milnor number, depending on (continuous) parameters, all of which yield inequivalent germs.