Recall:

<u>Lemma 4.7:</u> $f \in m^2_{R_n}$, $k = \mathrm{Corank}(f) \Longrightarrow \mu(f) \geq \frac{1}{2}k(k+1)$.

Consequence: $\mu(f) \leq 5 \Longrightarrow k \leq 2$

$\mathrm{Corank}(f) \in \{0,1\}$ already done!

remains: $\mu(f) \leq 5$, $\mathrm{Corank}(f) = 2$

splitting lemma: $f \underset{R}{\sim} g(x,y) + \sum\limits_{i>2}^{n} a_i x_i^2 \quad a_i \in \{1,-1\}$

$g \in m^3_{R_2}$. And: $\mathrm{Corank}(f) = 2 \underset{4.7.}{\Longrightarrow} \mu(f) \geq 4$.

so task: classify $g \in m^3_{R_2}$ s.t. $\mu(g) \in \{4,5\}$.

<u>we have</u> $g \underset{R}{\sim} g_3 + g_4 + \ldots + g_{\mu+1}$ with

$$g_d \in \mathbb{K}[x,y]_d := \left\{ \sum_{i+j=d} a_{ij} x^i y^j \;\middle|\; a_{ij} \in \mathbb{K} \right\}$$

(by finite determinacy)

cubic form

Theorem 4.8. (classification of cubic forms)

Let $g_3 = a \cdot x^3 + b \cdot x^2 y + c \cdot xy^2 + d \cdot y^3 \in \mathbb{K}[x,y]_3$ be given. Then there is $A \in Gl_2(\mathbb{K})$, s.t. for $\phi_A(x,y) := A \cdot \begin{pmatrix} x \\ y \end{pmatrix}$ (i.e. $\phi_A \in \mathcal{G}_2$) we have:

$$\phi_A^* g_3 \in \left\{ \begin{array}{c} 0 \\ x^3 \\ x^2 \cdot y \\ x^3 - xy^2 \\ x^3 + y^3 \end{array} \right\}$$

these forms are linearly inequivalent if $\mathbb{K} = \mathbb{R}$, if $\mathbb{K} = \mathbb{C}$, then only $x^3 - xy^2$ and $x^3 + y^3$ are equivalent

Proof: If one of these forms were equivalent (by lin. transformation) to another one from the list, then

The same must hold for:

1.) their vanishing loci

2.) their critical sets

germ/form	vanishing locus	critical set
0	$\mathbb{K}^2$	$\mathbb{K}^2$
x^3	$\{x=0\}$	$\{x=0\}$
$x^2 \cdot y$	$\{x=0\} \cup \{y=0\}$	$\{x=0\}$
$x^3 - x \cdot y^2$	$\{x=0\} \cup \{x=y\} \cup \{x=-y\}$	$\{0\}$
$x^3 + y^3$	$\{x=-y\}$ if $\mathbb{K}=\mathbb{R}$; if $\mathbb{K}=\mathbb{C}$: $\{x=-y\} \cup L_1 \cup L_2$, L_i 1-dim sub-vector spaces $\subset \mathbb{C}^2$	$\{0\}$

we see that for $\mathbb{K} = \mathbb{R}$ all five forms are inequivalent, for $\mathbb{K} = \mathbb{C}$ the last two ones could be equivalent (see below)

<u>Existence of A:</u> For $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Gl_2(\mathbb{K})$

we have $\phi_A^*(x) = \alpha x + \beta y$; $\phi_A(y) = \gamma x + \delta y$

$$\Rightarrow \phi_A^* g_3 = \phi_A^*\left(a \cdot x^3 + b \cdot x^2 y + c x y^2 + d y^3\right) =$$

$$a \cdot (\alpha x + \beta y)^3 + b(\alpha x + \beta y)^2(\gamma x + \delta y) + c(\alpha x + \beta y)(\gamma x + \delta y)^2$$

$$+ d(\gamma x + \delta y)^3 =$$

$$= a \cdot \alpha^3 x^3 + 3a\alpha^2\beta x^2 y + 3a\alpha\beta^2 x y^2 + a\beta^3 y^3$$

$$+ (b\alpha^2 x^2 + 2b\alpha\beta xy + b\beta^2 y^2)(\gamma x + \delta y)$$

$$(c\alpha x + c\beta y)(\gamma^2 x^2 + 2\gamma\delta xy + \delta^2 y^2)$$

$$+ d\gamma^3 x^3 + 3d\gamma^2\delta x^2 y + 3d\gamma\delta^2 x y^2 + d\delta^3 y^3$$

$$= a\cdot\alpha^3x^3 + 3a\alpha^2\beta x^2y + 3a\alpha\beta^2xy^2 + a\beta^3y^3$$
$$+ b\alpha^2\gamma x^3 + 2b\alpha\beta\gamma x^2y + b\beta^2\gamma xy^2 + b\alpha^2\delta x^2y + 2b\alpha\beta\delta xy^2 + b\beta^2\delta y^3$$
$$+ c\alpha\gamma^2x^3 + 2c\alpha\gamma\delta x^2y + c\alpha\delta^2xy^2 + c\beta\gamma^2x^2y + 2c\beta\gamma\delta xy^2 + c\beta\delta^2y^3$$
$$+ d\gamma^3x^3 + 3d\gamma^2\delta x^2y + 3d\gamma\delta^2xy^2 + d\delta^3y^3$$

$$= x^3\cdot\left(\overbrace{a\cdot\alpha^3 + b\cdot\alpha^2\gamma + c\alpha\gamma^2 + d\gamma^3}^{a'}\right)$$

$$+ x^2y\left(\overbrace{3a\cdot\alpha^2\beta + b\left(2\alpha\beta\gamma + \alpha^2\delta\right) + c\left(2\alpha\gamma\delta + \beta\gamma^2\right) + 3d\cdot\gamma^2\delta}^{b'}\right)$$

$$+ xy^2\left(\overbrace{3a\alpha\beta^2 + b\left(\beta^2\gamma + 2\alpha\beta\delta\right) + c\left(\alpha\delta^2 + 2\beta\gamma\delta\right) + 3d\gamma\delta^2}^{c'}\right)$$

$$+ y^3\left(\overbrace{a\beta^3 + b\beta^2\delta + c\beta\delta^2 + d\delta^3}^{d'}\right) = a'x^3 + b'x^2y + c'xy^2 + d'y^3$$

1. case: $a=0$: put $\alpha=\delta=0$ & $\beta=\gamma=1$

2. case: $a\neq 0$: put $\alpha=\delta=1$ & $\gamma=0$, $\beta\in\mathbb{K}$ s.t.

$a\beta^3+b\cdot\beta^2+c\cdot\beta+d=0$ (if $\mathbb{K}=\mathbb{R}$, such β exists in $\mathbb{R}$)

$\hookrightarrow \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in Gl_2(\mathbb{K})$ AND: $d'=0$

hence we can assume from the beginning that

$d=0$, i.e., $\boxed{g_3=ax^3+bx^2y+cxy^2}$ $\Leftarrow$

1. case: $c\neq 0$, put $\alpha=\delta=1$, $\beta=0$, $\gamma=-\frac{b}{2c}$

$b'=b\cdot\alpha^2\delta+c\cdot 2\alpha\gamma\delta=b+c\cdot 2\gamma\overset{\checkmark}{=}0$

hence $b'=0$

and $d'=0$ (since $d=0$ and $\beta=0$)

2. case: $c=0$, $b\neq 0$, put $\alpha=0$, $\beta=\gamma=1$, $\delta=-a/b$

$\Rightarrow b'=0$ (since $\alpha=0$, δ does not matter)

& $d'=0$ ($d'=a+b\cdot\delta \Rightarrow d'=0$ since $\delta=-a/b$)

3. case: $c=b=0$, put $\beta=0$ ($\alpha,\gamma,\delta=1$)

$\Rightarrow b'=d'=0$

+ (in all cases) $A=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in Gl_2(\mathbb{k})$

hence: can assume: $g_3(x,y)=a\cdot x^3+c\cdot xy^2$

Now $\exists$ 4 cases:

1. case: $a=c=0 \Rightarrow g_3=0$

2. case: $c=0$, $a\neq 0 \Rightarrow g_3=a\cdot x^3 \underset{\substack{\uparrow\\ \text{lin. equivalent}}}{\hat{=}} x^3$

3. case: $c\neq 0$, $a=0 \Rightarrow g_3=cxy^2 \hat{=} xy^2 \hat{=} x^2\cdot y$

4. case: $a \neq 0,\ c \neq 0$

$\mathbb{K} = \mathbb{C}$: $g_3 \triangleq x^3 - xy^2$

$\mathbb{K} = \mathbb{R}$: case 4.1.: $a \cdot c < 0 \Rightarrow g_3 \triangleq x^3 - xy^2$

$$\left(\text{put } x := \tilde{x} \cdot a^{-\frac{1}{3}} \Rightarrow g(\tilde{x}, y) = a \cdot \left(\tilde{x} a^{-\frac{1}{3}}\right)^3 + c\, \tilde{x} a^{-\frac{1}{3}} y^2 \right.$$

$$= \tilde{x}^3 + \frac{c}{a^{1/3}} \cdot \tilde{x} \cdot y^2$$

$$\left. \text{put } y = \underbrace{\left(-\frac{c}{a^{1/3}} \right)^{-1/2}} \cdot \tilde{y} \Rightarrow \ldots \Rightarrow g_3 = \tilde{x}^3 - \tilde{x}\tilde{y}^2 \right)$$

well-defined since $\frac{c}{a^{1/3}}$ is negative because $a \cdot c < 0$

case 4.2.: $a \cdot c > 0 \Rightarrow g_3 \triangleq x^3 + xy^2$

Put $x = \frac{1}{4^{1/3}} (\tilde{x} + \tilde{y})$; $y = \frac{3^{1/2}}{4^{1/3}} (\tilde{x} - \tilde{y})$

$$\Rightarrow g_3 = \dots\dots = \tilde{x}^3 + \tilde{y}^3$$

$$g_3 \hat{=} x^3 + y^3$$

$\square$

<u>Corollary 4.9:</u> Let $f, g \in \{0, x^3, x^2y, x^3+y^3, x^3-xy^2\} \subset \mathcal{R}_2$. Then if $f \neq g$, we have

$$\begin{cases} f \underset{R}{\not\sim} g & \text{if } \mathcal{R}_2 = \mathcal{E}_2 \\ x^3+y^3 \underset{R}{\sim} x^3-xy^2, \; f \underset{R}{\not\sim} g \text{ in all other cases} & \text{if } \mathcal{R}_2 = \mathcal{O}_2 \end{cases}$$

Moreover: $f \in \{0, x^3, x^2y\} \subset \mathcal{R}_2$ then $\mu(f) = \infty$, f not finite determined

$f \in \{x^3+y^3,\ x^3-xy^2\} \Rightarrow \mu(f) = 4,$

determinacy $(f) = 3.$

<u>Proof</u>: Suppose $f \underset{R}{\sim} g$, i.e. $\exists\, \varphi \in \mathcal{G}_2 : \varphi^* f = g$

Put $\varphi = \varphi_1 + \varphi_{>1}$ with $\varphi_1 \in Gl_2(\mathbb{K})$ and

$\varphi_{>1} \in \left(\mathfrak{m}^2_{R_2}\right)^{\oplus 2}$. Since $f, g \in \mathbb{K}[x,y]_3$

$\Rightarrow \varphi_1^* f = g$ ↯ Th. 4.8.

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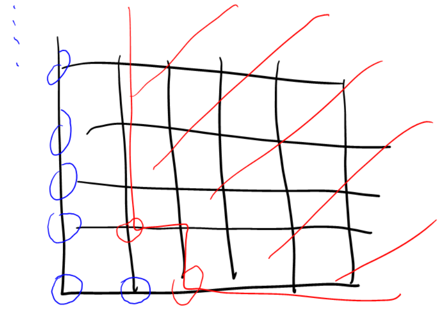
clear: $\mu(0) = \mu(x^3) = \mu(x^2y) = \infty$

$\Big($e.g. $\mu(x^2y) = \dim_{\mathbb{K}} R_2 / \left(\partial_x(x^2y), \partial_y(x^2y)\right)$

$= \dim R_2 / (2xy, x^2) = \infty$



$\Big)$

also $\mu(x^3+y^3) = \mu(x^3-xy^2) = 4$

already seen (ex. 4.+5. after Lemma 3.8):

determinacy of x^3+y^3, $x^3-xy^2 = 3$ $\square$

Now starts classification of germs

with $\mu \leq 5$ up to stable right equivalence

<u>Theorem 4.10:</u> Let $f \in m^2_{\mathbb{R}_n}$, $\operatorname{Corank}(f)=2$,

$\mu(f)=4$, then f is stably equivalent

to either x^3-xy^2 or x^3+y^3 (both are

equivalent for $\mathbb{R}_n = \mathcal{O}_n$).