§4. Classification

<u>Aim</u>: Classify germs $f \in R_n$ with $\mu(f) \leq 5$ up to right equivalence.

<u>Tool</u>: Corank of a germ.

<u>Def. 4.1.</u>: Let $f \in m^2_{R_n} \subset R_n$, then we call the number $n - \mathrm{rk}(D^2 f)(0)$ the <u>corank</u> of f.

<u>Rk</u>: Lemma 3.9 + Th. 1.8. (Morse lemma) tells us: $\mathrm{Corank}(f) = 0 \Longleftrightarrow f$ is non-degenerate at $0 \Longleftrightarrow \mu(f) = 1$.

If $\mathrm{Corank}(f) > 0$, then f is not right equivalent to quadratic form. But we have the following generalization of Morse lemma:

Th. 4.2.: Let $f \in m^2_{R_n}$, $k := \operatorname{corank}(f)$

Then $\exists g \in m^3_{R_k} : \exists a_{k+1}, \dots, a_n \in \{1, -1\}$

s.t. $\overline{a_{k+1}, \dots, a_n} \in \begin{cases} \{1,-1\} & R_n = \mathcal{E}_n \\ \{1\} & R_n = O_n \end{cases}$ s.t.

$$f \underset{R}{\sim} g + a_{k+1} \cdot x_{k+1}^2 + \dots + a_n \cdot x_n^2$$

(Splitting lemma)

Proof: (assume for simplicity that f is finitely determined) $\leftarrow$

obviously, we can write (since $\overset{k}{\overset{\shortparallel}{\operatorname{corank}(f)}}$)

$$f \sim f_3(x_1, \dots, x_n) + \overbrace{a_{k+1} x_{k+1}^2 + \dots + a_n x_n^2}^{q}$$

where $f_3 \in m^3_{\mathbb{R}_n}$

we can write:

$$f_3(x_1,\dots,x_n) = g_3(x_1,\dots,x_k) + \sum_{j=k+1}^{n} x_j \cdot \underbrace{h_j^{(2)}}_{\in m^3_{\mathbb{R}_n}}$$

where $g_3 \in m^3_{\mathbb{R}_k}$, $h_j^{(2)} \in m^2_{\mathbb{R}_n}$

hence: $f \sim q + \sum_{j=k+1}^{n} x_j \cdot h_j^{(2)} + g_3$

$$= \sum_{j=k+1}^{n} \left[a_j \cdot \Big(\underbrace{x_j + \frac{a_j}{2} h_j^{(2)}}_{Y_j}\Big)^2 - \frac{1}{4}\left(h_j^{(2)}\right)^2 \right] + g_3$$

Define coordinate change:

$$y_i := \phi^*(x_i) = \begin{cases} x_i & i \in \{1, \dots, k\} \\ x_i + \frac{a_i}{2} h_i^{(2)} & i \in \{k+1, \dots, n\} \end{cases}$$

$$(D\phi)(0) = \mathrm{Id} \Longrightarrow \phi \in \mathcal{G}_n \Longrightarrow \exists \psi := \phi^{-1}$$

$$\psi^*\left(q + \sum_{j=k+1}^{n} x_j h_j^{(2)} + g_3\right)$$

$$= g_3(y_1, \dots, y_k) + \sum_{j=k+1}^{n} a_j \cdot y_j^2$$

$$\underbrace{- \frac{1}{4} \sum_{j=k+1}^{n} \left(h_j^{(2)}\right)^2 \circ \psi}_{\overset{!!}{=}\; f_4 \,\in\, m_{\mathcal{R}_n}^4} \qquad \underset{R}{\sim} f$$

again: $f_4 = \underbrace{g_4(y_1, \dots, y_k)}_{\in m^4_{\mathbb{R}_k}} + \sum_{j=k+1}^{n} y_j \cdot h_j^{(3)}$

new transformation:

$$f \sim g_3 + g_4 + q + \overbrace{f_5}^{\in m^5_{\mathbb{R}_n}}$$

$\vdots$

$$\boxed{f \sim \sum_{i=3}^{N} g_i + q + f_{N+1}}$$

with $g_i \in m^i_{\mathbb{R}_k}$, $f_{N+1} \in m^{N+1}_{\mathbb{R}_n}$

Put $g := \sum_{i=1}^{N} g_i$, Now assume that

f is N-determined $\Longrightarrow f \sim g + q$

$g \in m^3_{\mathbb{R}_k}$ and $q = \sum_{j=1}^{n} a_j \cdot x_j^2$, $a_j \in \{1, -1\}$ $\square$

Def. 4.3: Let $f \in \mathcal{R}_n$, $g \in \mathcal{R}_k$, $k \leq n$

Then f and g are called stably right equivalent, iff

$$f(x_1, \ldots, x_n) \underset{R}{\sim} g(x_1, \ldots, x_k) + a_{k+1} x_{k+1}^2 + \ldots + a_n x_n^2$$

where $a_i \in \begin{cases} \{1, -1\} & \mathcal{R}_n = \mathcal{E}_n \\ \{1\} & \mathcal{R}_n = \mathcal{O}_n \end{cases}$

(modified) aim: classify germs (up to $\mu \leq 5$) w.r.t. stable right equivalence.

Lemma 4.4: Let $f \in \mathcal{R}_n$, $g \in \mathcal{R}_m$

where q is non-degenerate quadratic form. Then $\mu(f+q) = \mu(f)$.

Proof: Put $r := m+n$, so that $f+q \in R_r$,

then $J_{f+q} = \underline{J_f \cdot R_r} + \underline{J_q \cdot R_r}$, but

$J_q = m_{R_m} \cdot R_r$ and $\underline{R_r = m_{R_m} R_r \oplus R_n}$

$$\Rightarrow J_{f+q} \underset{\Gamma}{=} \boxed{J_f} + \boxed{m_{R_m} R_r}$$

as $\mathbb{K}$-vect. spaces

$$\Rightarrow \underbrace{R_r / J_{f+}}_{} \simeq \underbrace{R_n / J_f}_{}$$

$\mu(f+q) = \mu(f)$ $\square$

Lemma 4.5.: $f \in m^2_{\mathcal{R}_n}$, $\mu(f) < \infty$

Corank$(f) = 1$, then f is stably equivalent to $\varepsilon \cdot x^{\mu+1}$, $\varepsilon \in \begin{cases} \{1,-1\} & \mathcal{R}_n = \mathcal{E}_n \\ \{1\} & \mathcal{R}_n = \mathcal{O}_n \end{cases}$

(+ $\varepsilon = 1$ if $\mathcal{R}_n = \mathcal{E}_n$ & n even).

<u>Proof</u>: From Th. 4.7:

$$\boxed{f \underset{R}{\sim} g(x_1) + q(x_2, \ldots, x_n)}$$

where q is a non-deg. quadratic form & $\mu(f) = \mu(g)$ (4.4.)

Lemma 1.6. $\Longrightarrow g(x_1) \sim \varepsilon x_1^{\mu(g)+1}$

$= \varepsilon \cdot x_1^{\mu+1}$ $\Rightarrow f$ stably equ. to $\varepsilon \cdot x_1^{\mu+1}$ □

next step: Corank ≥ 2.

we need an estimation $\mu \leftrightarrow$ Corank:

<u>Lemma 4.6.</u>: Let $f \in m^2_{R_n}$, $\mu(f) < \infty$

then $J_f = m_{R_n} \cdot J_f \oplus \mathbb{K}\partial_{x_1} f \oplus \ldots \oplus \mathbb{K}\partial_{x_n} f$

<u>Pf.</u>: clear: $m_{R_n} \cdot J_f \subset J_f$ and $\partial_{x_i} f \subset J_f$

show "$\subseteq$": Let $g \in J_f$, i.e.:

$$\exists \lambda_1, \ldots, \lambda_n \in R_n : g = \sum_{i=1}^{n} \lambda_i \cdot \partial_{x_i} f$$

put $\tilde{\lambda}_i := \lambda_i - \underbrace{\lambda_i(0)}_{\in \mathbb{K}} \Rightarrow \tilde{\lambda}_i \in m_{R_n}$

(since $\tilde{\lambda}_i(0) = 0$)

$$\Rightarrow g = \sum_{i=1}^{n} \tilde{\lambda}_i \cdot \partial_{x_i} f + \sum_{i=1}^{n} \lambda_i(0) \cdot \partial_{x_i} f \in mJ_f + \sum \mathbb{K}\partial_{x_i} f$$

Now we have shown:

$$\mathfrak{d}_f = \underbrace{m_{\mathfrak{R}_n}\mathfrak{J}_f + \mathbb{K}\partial_{x_1}f + \ldots + \mathbb{K}\partial_{x_n}f}$$

still to show: this sum is direct.

<u>Claim</u>: sufficient to show:

$$\sum_{i=1}^{n} h_i(x)\cdot(\partial_{x_i}f)(x) = 0$$

$$\Rightarrow h_i(0) = 0$$

<u>since</u>: $\sum g_i(x)\,\partial_{x_i}f + \sum a_i\,\partial_{x_i}f = 0$; $g_i \in m_{\mathfrak{R}_n}$, $a_i \in \mathbb{K}$

$$\Longrightarrow \underset{\overset{\shortparallel}{0}}{g_i(0)} + a_i = 0 \Rightarrow a_i = 0$$

$$\Rightarrow \sum g_i(x)\cdot\partial_{x_i}f = 0$$

show claim: Assume it doesn't

hold $\Longrightarrow \exists \varphi \in G_n : \varphi^*\left(\sum_{i=1}^{n} h_i(x)\, \partial_{x_i} f\right)$

(say $h_1(0) \neq 0$)

$= \partial_{x_1}(f \circ \varphi)$ uses existence of ODE as in pf of 3.6.

Hence $\partial_{x_1}(f \circ \varphi) = 0 \Longrightarrow f \circ \varphi_1 \in R_{n-1} \subset R_n$

$\mu(f) = \infty$ ↯

Lemma 4.7.: $f \in m^2_{R_n}$, $k = \operatorname{Corank}(f)$

then $\mu(f) \geq \frac{1}{2} k (k+1)$

(possibly $\mu(f) = \infty$)

(e.g. $\mu(f) \leq 5 \Longrightarrow k \leq 2$)

Pf.: 1. case: $k=0$ then by Morse lemma: $\mu(f)=1 \Rightarrow 1 > \underbrace{\frac{1}{2}\,0\cdot 1}_{\substack{\shortparallel \\ 0}}$ ✓

2. case: $k>0$, $\mu(f)=\infty$: ✓

3. case: $k>0$, $\mu(f)<\infty$:

Th. 4.2. $\Rightarrow f \sim \underbrace{g(x_1,\dots,x_k)}_{\in m^3_{R_k}} + \underbrace{q(x_{k+1},\dots,x_n)}_{\substack{\text{non-deg.} \\ \text{quadratic form}}}$

$\xrightarrow{4.4.} \underline{\mu(f)=\mu(g)}$

Put $I := \mathfrak{m}_{R_k} \cdot J_g \subset R_k$. Apply

Lemma 4.6. to g $\left(J_g = \underbrace{\mathfrak{m}_{R_k} J_g}_{I} \oplus \underbrace{\mathbb{K}\partial_{x_1} g \oplus \dots \oplus \mathbb{K}\partial_{x_k} g}_{k\text{-dim}}\right)$

$$\mu(g) = \dim_{\mathbb{K}} R_k / J_g = \dim_{\mathbb{K}} R_k / I - k$$

$$g \in \mathfrak{m}^3_{R_k} \implies J_g \subset \mathfrak{m}^2_{R_k} \implies I \subset \mathfrak{m}^3_{R_k}$$

$\exists$ surjection $R_k / I \twoheadrightarrow R_k / \mathfrak{m}^3_{R_k}$

$$\implies \dim_{\mathbb{K}} R_k / I \geq \dim_{\mathbb{K}} R_k / \mathfrak{m}^3_{R_k} = \binom{k+2}{2}$$

$$\implies \mu(g) \geq \binom{k+2}{2} - k > \frac{(k+2)(k+1)}{2} - k - 1$$

$$= \dots = \frac{k(k+1)}{2} \qquad \square$$