Singularity theory – Sheet 8

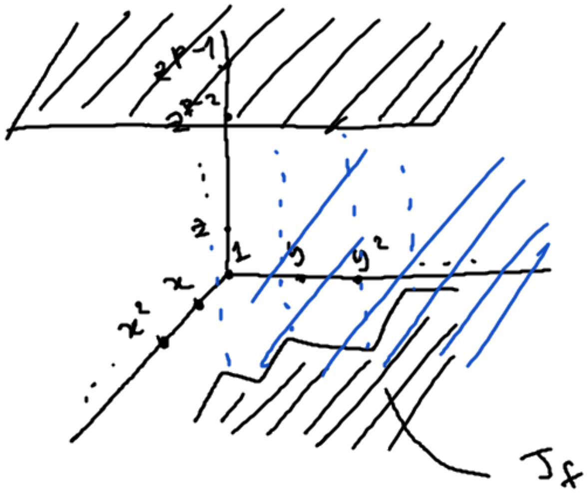
(1) $f \in R_2 \rightsquigarrow g(x,y,z) := z^p + f(x,y) \in R_3$, $p \in \mathbb{N}$ $(p \geq 2)$

$Dg = (\underbrace{\partial_x f, \partial_y f}_{Df}, p\, z^{\overset{\geq 1}{p-1}}) \rightsquigarrow J_g = (\underbrace{\partial_x f, \partial_y f}_{J_f}, z^{p-1})$

when $p=1$, $J_g \ni 1 \Rightarrow J_g = R_3$

$\mu(g) = \dim R_3/J_g = 0$

z^{p-1} is surely independent of $\partial_x f, \partial_y f$



$$\mu(g) = \dim_{\mathbb{C}} \left(R_3/J_g \right)$$

$$= \mu(f) \cdot (p-1)$$

(2) $R_n \supseteq \mathcal{A}$ ideal.

(1) $\dim_k R_n/\mathcal{A} < +\infty \iff \exists\, k \geq 1$ s.th. $\underbrace{\mathfrak{m}_{R_n}}_{\mathfrak{m}}{}^{k} \subseteq \mathcal{A}$:

$\Leftarrow$: $\mathfrak{m}^k \subseteq \mathcal{A}$: $\quad \dim_k R_n/\mathcal{A} \leq \dim_k R_n/\mathfrak{m}^k \underset{\uparrow\ \text{Lecture 6}}{=} \dim_k k[x_1, \dots, x_n]_{\leq k} < +\infty$

$\Rightarrow$: $\dim_k R_n/\mathcal{A} =: a_0 < \infty$

$$R_n \supseteq \mathcal{A} + \mathfrak{m} \supseteq \mathcal{A} + \mathfrak{m}^2 \supseteq \dots \supseteq \mathcal{A} + \mathfrak{m}^{j-1} \supseteq \mathcal{A} + \mathfrak{m}^j \supseteq \dots \supseteq \mathcal{A}$$

let be $\dim_k \left((\mathcal{A} + \mathfrak{m}^j)/\mathcal{A} \right) =: a_j$.

$$\underset{\overset{\|}{a_0}}{\dim_k \left(R_n/\mathcal{A} \right)} \geq \underset{\overset{\|}{a_1}}{\dim_k \left((\mathcal{A} + \mathfrak{m})/\mathcal{A} \right)} \geq \dots \geq \underset{\overset{\|}{a_j}}{\dim_k \left((\mathcal{A} + \mathfrak{m}^j)/\mathcal{A} \right)} \geq \dots$$

$\rightsquigarrow$ we get a sequence $\infty > a_0 \geq a_1 \geq a_2 \geq \dots \geq a_j \geq \dots \geq 0$

$\Rightarrow \exists\, k \geq 1$ s.th. $\mathcal{A} + \mathfrak{m}^k = \mathcal{A} + \mathfrak{m}^{k+1} \xRightarrow{\text{Nakayama}} \mathfrak{m}^k \subseteq \mathcal{A}$.

Nakayama: B f.g. s.th. $B \subseteq C + mB \Rightarrow B \subseteq C$.

take: $B = m^k$ (f.g), $C = \mathcal{A}$.

Then: $m^k \subseteq \mathcal{A} + m^k = \mathcal{A} + m^{k+1} = \mathcal{A} + m \cdot m^k \overset{\text{Nak.}}{\Longrightarrow} m^k \subseteq \mathcal{A}$.

In particular: $\mathcal{A} = J_f$. Given f, $\mu(f) < \infty \Leftrightarrow \exists k \geq 1$ s.th. $m^k \subseteq J_f$.

(2) Assume $0 < \dim(R_n/\mathcal{A}) < \infty$; then: $\exists k, l \geq 1$ s.th. $m^k \subseteq \mathcal{A} \subseteq m^l$ and

$$m^{k-1} \not\subseteq \mathcal{A} \not\subseteq m^{l+1}.$$

By part (a): $\exists k \geq 0$ s.th. $m^k \subseteq \mathcal{A}$, and we take the smallest $k \geq 1$ s.th. this happens.

[$\dim R_n/\mathcal{A} > 0 \Rightarrow \mathcal{A} \neq R_n = m^0 \Rightarrow \exists k' \geq 0$ s.th $m^{k'} \not\subseteq \mathcal{A}$.]
$(m^0 \not\subseteq \mathcal{A})$

$\mathcal{A} \neq R_n \Rightarrow \forall f \in R_n$ s.th $f(0) \neq 0$, $f \notin \mathcal{A}$ (such f is invertible)

$\Rightarrow \mathcal{A} \subseteq m$.

To show: $\exists$ a largest $\ell \geq 1$ s.th $\mathfrak{A} \subseteq m^{\ell}$.

If it was not true: $\mathfrak{A} \subset \bigcap_{\ell=0}^{\infty} m^{\ell} = m^{\infty}$

Taylor morphism

$$\Rightarrow \dim R_n/\mathfrak{A} \geq \dim R_n/m^{\infty} = \dim \mathrm{Im}(T) = +\infty \quad \lightning .$$

$k = \mathbb{C}$: $\dim_{\mathbb{C}} \mathbb{C}\{x_1, \dots, x_n\}$

$k = \mathbb{R}$: $\dim_{\mathbb{R}} \mathbb{R}[[x_1, \dots, x_n]]$

(c) $\ell \geq 0$. Show: $m^{\ell} \subseteq R_1$ is the unique ideal $\mathfrak{A}$ satisfying $\dim_k R_1/\mathfrak{A} = \ell$.

* $\dim_k R_1/m^{\ell} = \dim_k k[x]_{\underset{*}{<} \ell} = \ell$.
* is m^{ℓ} the only ideal satisfying $\dim_k R_1/\mathfrak{A} = \ell \geq 0$?
  - $\ell = 0$: $\mathfrak{A} = R_1 = m^0$ ✓

• $\underset{\dim_k R_1/\mathcal{I}}{\overset{\shortparallel}{\ell}} \geq 1$: $\qquad \mathcal{I} \neq R_1 \underset{\text{part (b)}}{\overset{}{\Longrightarrow}} \exists$ a maximal $k \geq 1$ s.th. $\mathcal{I} \subseteq m^k$
$(\mathcal{I} \not\subseteq m^{k+1})$

$\Rightarrow \exists f \in \mathcal{I}$ germ which is not in m^{k+1}

$\underset{\text{Taylor}}{\Rightarrow} f = \underset{\text{generator of } m^k}{x^k} (\underbrace{1+g}_{\text{inv}})$, $g \in m$

$\Rightarrow \underbrace{(f)}_{\text{ideal gen. by } f \text{ in } R_1} \overset{\subseteq \mathcal{I}}{=} m^k \quad \Rightarrow \quad m^k \subseteq \mathcal{I}$

$\Rightarrow \mathcal{I} = m^k$.

(3) (a) $f \in R_n$ with $\det(f) = d \Rightarrow \mu(f) \leq \binom{n+d}{n}$.

$f \in R_n$ is determined $\Rightarrow \exists k \in \mathbb{N}$ s.th. $m^k \subseteq J_f$. (by lectures).

By ex 2.(b), $\exists$ a minimal $k \geq 0$ s.th. $m^k \subseteq J_f$.

not necessary

Thm 3.10 lecture 9 : f d-determined $\Rightarrow m^{d+1} \subset m J_f$.

But $m J_f \subset J_f \Rightarrow m^{d+1} \subseteq J_f$. Then:

$$\mu(f) = \dim_k R_n / J_f \leq \dim_k R_n / m^{d+1} = \dim k[x_1, \ldots, x_n]_{\leq d} = \binom{n+d}{n}.$$

(b) $f \in m^k$, $\ell \in \mathbb{N}$ $\Rightarrow$ $\mu(f) \geq \binom{n+k+\ell-1}{k+\ell-1} - n\binom{n+\ell}{\ell}$.

<u>Hint</u>: $\forall \ell \in \mathbb{N}$, $J_f + m^{k+\ell} = R_n^{\leq \ell}\partial_{x_1} f + \dots + R_n^{\leq \ell}\partial_{x_n} f + m^{k+\ell}$

where: $R_n^{\leq \ell}\partial_{x_1} f \ni x^\nu \partial_{x_1} f$, $0 \leq |\nu| \leq \ell$.

<u>Claim</u>: $\mathcal{A} \subseteq R_n$ ideal, $\mathcal{A} = \langle g_1, \dots, g_r \rangle$, $g_i \in R_n$. Then:

$$\mathcal{A} = m^{\ell+1}\cdot\mathcal{A} + R_n^{\leq \ell} g_1 + \dots + R_n^{\leq \ell} g_r.$$

- <u>Claim $\Rightarrow$ Hint</u>:

$J_f = \langle \partial_{x_1} f, \dots, \partial_{x_n} f \rangle \overset{\text{claim}}{\Longrightarrow} J_f = m^{\ell+1} J_f + R_n^{\leq \ell}\partial_{x_1} f + \dots + R_n^{\leq \ell}\partial_{x_n} f$.

$\Rightarrow J_f + m^{k+\ell} = \underbrace{m^{\ell+1} J_f + m^{k+\ell}}_{= m^{k+\ell}} + R_n^{\leq \ell}\partial_{x_1} f + \dots + R_n^{\leq \ell}\partial_{x_n} f$

because $m^{\ell+1} J_f \subseteq m^{k+\ell}$, since $f \in m^k \Rightarrow J_f \subseteq m^{k-1}$.

- Hint ⇒ exercise:

we have: $J_f + \mathfrak{m}^{k+l} = R_n^{\le l}\partial_{x_1} f + \_\_ + R_n^{\le l}\partial_{x_n} f + \mathfrak{m}^{k+l}$, and

$J_f \subseteq J_f + \mathfrak{m}^{k+l}$. Then:

$$\mu(f) = \dim_k R_n/J_f \ge \dim_k R_n/_{J_f + \mathfrak{m}^{k+l}} =$$

$$= \dim_k \left(\frac{R_n/\mathfrak{m}^{k+l}}{J_f + \mathfrak{m}^{k+l}/\mathfrak{m}^{k+l}} \right) =$$

double quot thm; $A \subseteq B \subseteq C$, $A/B \cong {}^{A/C}/_{B/C}$

$$= \underbrace{\dim_k\left(R_n/\mathfrak{m}^{k+l}\right)}_{\cong k[x_1,\dots,x_n]_{\le k+l-1}} - \underbrace{\dim_k\left(\frac{R_n^{\le l}\partial_{x_1}f + \_\_ + R_n^{\le l}\partial_{x_n}f + \mathfrak{m}^{k+l}}{\mathfrak{m}^{k+l}}\right)}_{\le \dim_k\left(R_n^{\le l}\partial_{x_1}f + \_\_ + R_n^{\le l}\partial_{x_n}f\right)}$$

$\dim_k\left(R_n/\mathfrak{m}^{k+l}\right) = \binom{n+k+l-1}{k+l-1}$

$A \twoheadrightarrow \frac{A+B}{B}$

$\dim_k\left(R_n^{\le l}\partial_{x_1}f + \_\_ + R_n^{\le l}\partial_{x_n}f\right) = n\binom{n+l}{l}$

$$\ge \binom{n+k+l-1}{k+l-1} - n\binom{n+l}{l}.$$

- Proof of claim: $\mathfrak{a} = \langle g_1, \dots, g_r \rangle$, $g_i \in R_n$. Then: $\mathfrak{a} = \mathfrak{m}^{\ell+1}\mathfrak{a} + R_n^{\leq \ell} g_1 + \dots + R_n^{\leq \ell} g_r$. $\forall \ell \geq 1$.

given $h_j \in R_n$:

$$\mathfrak{a} \ni \sum_{j=1}^{r} g_j h_j = \underbrace{\sum_{j=1}^{r} g_j \underbrace{(h_j - h_j(0))}_{\mathfrak{m}}}_{\mathfrak{m}\mathfrak{a}} + \underbrace{\sum_{j=1}^{r} g_j \overbrace{h_j(0)}^{\in k}}_{\in \underbrace{k g_1 + \dots + k g_r}_{R_n^{\leq 0} g_1 + \dots + R_n^{\leq 0} g_r}}$$

$$\Rightarrow \mathfrak{a} \overset{(*)}{=} \mathfrak{m}\mathfrak{a} + R_n^{\leq 0} g_1 + \dots + R_n^{\leq 0} g_r.$$

Apply this to $\mathfrak{m}\mathfrak{a}$: $\mathfrak{m}\mathfrak{a} = \mathfrak{m}^2\mathfrak{a} + R_n^{=1} g_1 + \dots + R_n^{=1} g_r$, and then

$(*)$: $\mathfrak{a} = \mathfrak{m}^2\mathfrak{a} + R_n^{\leq 1} g_1 + \dots + R_n^{\leq 1} g_r$, which is the final formula when $\ell = 1$.

induction $\rightsquigarrow$ $\mathfrak{a} = \mathfrak{m}^{\ell+1}\mathfrak{a} + R_n^{\leq \ell} g_1 + \dots + R_n^{\leq \ell} g_r$.