Sylvester's law of inertia:

$h : V \times V \longrightarrow \mathbb{R}$ basis for $V = \{v_1, \_\_\_, v_n\}$ $n = \dim_{\mathbb{R}} V$

$$\Big\downarrow$$

$$H = \begin{pmatrix} h(v_1,v_2) & h(v_1,v_2) & \\ & \cdots & \\ & & \end{pmatrix} \in M_{n \times n}(\mathbb{R})$$

$H_{ij} = h(v_i, v_j)$

To show: $\exists$ basis $\{v_1, \_\_, v_n\}$ of V s.th.

$$H = \begin{pmatrix} 1 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & -1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & -1 & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ 0 & & & & & & & & 0 \end{pmatrix} \quad \in \{1, -1, 0\}$$

Taylor development/homomorphism

$f \in R_n$

— In one variable:

$f = f(x) : \mathbb{R} \longrightarrow \mathbb{R}$

Taylor developement of f around 0 : $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots \Big|^{k}$

$$T^k(f) = \underbrace{f(0)}_{\in \mathbb{R}} + \underbrace{f^{(1)}(0)}_{\in \mathbb{R}} x + \dots + \underbrace{f^{(k)}}_{\in \mathbb{R}} \frac{(0)x^k}{k!}$$

$$f \longmapsto T(f) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \in \mathbb{R}[[x]]$$

$$T : \mathcal{E}_1 \longrightarrow \mathbb{R}[[x]]$$
$$f \longmapsto T(f)$$

$$T(f+g) = T(f) + T(g)$$

In more variables:

R_n : $f: \mathbb{R}^n \to \mathbb{R}$

$$D^v f := \frac{\partial^{v_1}}{\partial x_1} \cdots \frac{\partial^{v_n}}{\partial x_n} f$$

$v \in \mathbb{N}^n$

$v = (v_1, \ldots, v_n)$

$$T : R_n^{2m} \longrightarrow k[[x_1, \ldots, x_n]]$$

$$f \longmapsto \sum_{v \in \mathbb{N}^n} \underbrace{D^v f|_0}_{\in k} \underline{x}^v$$

Taylor morphism

$\ker(T) = m^\infty = \bigcap_{k \geq 1} m^k$

$$\underline{x}^v = x_1^{v_1} \ldots x_n^{v_n}$$

$$T^k(f) = \sum_{\substack{v \in \mathbb{N}^n \\ |v| \leq k}} D^v f|_0 \underline{x}^v \in k[x_1, \ldots, x_n]$$

$|v| = v_1 + \ldots + v_n$