Singularity theory - Ex. class 5

(1) R, $1 \in R$ neutral element, $m \subseteq R$.

(a) $\forall x \in m \Rightarrow 1+x$ is invertible in R.

<u>Recall</u>: (R, m) is local $\Leftrightarrow$ $m =$ ideal of the non invertible elements in R

Then; we need to show that $1+x \notin m$.

But: $1 \notin m$, $x \in m$; $1+x \in m \Rightarrow \underbrace{(1+x)}_{m} - \overbrace{x}^{m} \in m$

$\Rightarrow 1 \in m$ ↯.

(b) $I \subseteq \overset{\text{local}}{R}$ ideal $\Rightarrow R/I$ is local. ($I \subsetneq R$ proper)

There is a corrispondence (inclusion preserving):

$$\{\text{ideals of } R/I\} \underset{\pi^*}{\overset{\pi_*}{\underset{1:1}{\longrightarrow}}} \{\text{ideals of } R \text{ containing } I\}$$

$\pi : R \to R/I$ ↗

Then:

* $m \subseteq R$ max $\Rightarrow I \subseteq m$
  $\rightsquigarrow m/I$ max in R/I
* is the only one.

(c) $R = \mathbb{R}[[x_1, \ldots, x_n]] \ni \sum_{\nu \in \mathbb{N}^n} a_\nu \underline{x}^\nu = \alpha$ $\qquad \nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{x}^\nu = x_1^{\nu_1} \cdots x_n^{\nu_n}$

$\cup|$

$\mathfrak{m} = \{ \alpha$ as before s.th. $a_0 = 0 \}$ ($\leftarrow$ ex 2.a)

For any $\alpha \in \mathfrak{m}$, find an explicit inverse for $1+\alpha$.

$1 + \alpha = \sum_{\nu \in \mathbb{N}^n} a_\nu x^\nu$, where $a_0 = 1$.

$\beta = \sum_{\nu \in \mathbb{N}^n} b_\nu x^\nu$ is an inverse of $1+\alpha \iff \beta(1+\alpha) \overset{(*)}{=} 1$. We find β by induction.

$\cap$

R

$$(*) = \Big(\sum_{\nu \in \mathbb{N}^n} b_\nu x^\nu\Big)\Big(\sum_{\nu \in \mathbb{N}^n} a_\nu x^\nu\Big) = \sum_{\nu'} \Big(\sum_{\mu \le \nu'} a_\mu b_{\nu'-\mu}\Big) x^{\nu'} \overset{!}{=} 1$$

then: (1) $\underset{1}{a_0} b_0 = 1$ ok when $b_0 = 1$.

(2) $\sum_{\mu \le \nu} a_\mu b_{\nu-\mu} = \underset{1}{a_0} b_\nu + \sum_{0 < \mu \le \nu} a_\mu b_{\nu - \mu} \overset{!}{=} 0 \iff b_\nu = -\sum_{0 < \mu \le \nu} a_\mu b_{\nu-\mu}$ given by the previous ones.

(d) $P = \mathbb{R}[x_1, \dots, x_n]$ is not local:

$$\begin{array}{ccc} P & \xrightarrow{ev_0} & \mathbb{R} \\ p(\underline{x}) & \longmapsto & p(0) \end{array}$$

- is surjective: $\forall \alpha \in \mathbb{R}$, $ev_0(\alpha) = \alpha$.
- $\ker(ev_0) = (x_1, \dots, x_n) = m$

$P / \underbrace{\ker(ev_0)}_{m} \cong \mathbb{R}$ a field $\Rightarrow$ m is maximal.

(a) $\Rightarrow$ $1 + \overset{m}{\overset{\in}{x_1}} \notin m$, but $1 + x_1$ is not invertible in P:

$(1 + x_1) q(\underline{x}) \neq 1$ because $\deg((1 + x_1) q(\underline{x})) = 1 + \deg(q(\underline{x})) \geq 1$ while $\deg(1) = 0$.

(e) $(R, \mathfrak{m}) \rightsquigarrow R/\mathfrak{m}$ = residue class field.

Show that $\mathcal{E}/\mathfrak{m} \cong \mathbb{R}$:

$$\begin{aligned} \mathcal{E} &\xrightarrow{ev_0} \mathbb{R} \\ f &\longmapsto f(0) \end{aligned}$$

well def because $f = g$ in $\mathcal{E}$ $\Leftrightarrow$ they coincide around 0

* ev_0 is surjective ($f \equiv \alpha$ constant $\rightsquigarrow$ $f(0) = \alpha$)
* $\operatorname{Ker}(ev_0) = \{ f \in \mathcal{E} \mid f(0) = 0 \} = \mathfrak{m}$

$\Rightarrow \mathcal{E}/\mathfrak{m} \cong \mathbb{R}$.

(d) $(R, \mathfrak{m}) \rightsquigarrow H_R : \mathbb{N} \to \mathbb{N}$, $H_R(d) = \dim_k \left(\mathfrak{m}^d / \mathfrak{m}^{d+1} \right)$, $k = R/\mathfrak{m}$.

[$\mathfrak{m}^d/\mathfrak{m}^{d+1}$ is an R-mod, but also a $\overset{k}{R/\mathfrak{m}}$-mod $= k$-v.sp.]

(i) Take $R = \mathbb{R}[[x_1, \dots, x_n]]$;

$\mathfrak{m} = (x_1, \dots, x_n)$; what is $\mathfrak{m}^k / \mathfrak{m}^{k+1}$? ($k = \mathbb{R}$)

$\mathfrak{m}^k$ = ideal $\underbrace{\text{gen.}}_{\text{of } R}$ by homogeneous pol. of deg $\underline{\underline{k}}$ in $x_1, \dots, x_n$

$\mathfrak{m}^k / \mathfrak{m}^{k+1}$ = generated on $\underline{\underline{\mathbb{R}}}$ by homog. pol. of deg k in $x_1, \dots, x_n$ (this is a basis)

$\rightsquigarrow \mathfrak{m}^k / \mathfrak{m}^{k+1} = \{$ pol. of deg k in $x_1, \dots, x_n$ with coeff in $\mathbb{R}\}$

$\rightsquigarrow \dim_{\mathbb{R}} \mathfrak{m}^k / \mathfrak{m}^{k+1} = \binom{n+k-1}{k}$.

take $R = \mathcal{E}_n$:

$m = \{ f \in R \mid f(0) = 0 \} \rightsquigarrow f(\underline{x}) = \sum a_i x_i + \sum_{i,j} g_{ij}(\underline{x}) x_i x_j$

Lemma during one lecture

$\rightsquigarrow m/m^2$ is generated by x_i on $\mathbb{R}$

$\rightsquigarrow m^k/m^{k+1}$ is gen. by hom. pol. of deg k in $x_1, \dots, x_k$. If they are a basis, $H_R(k) = \binom{n+k-1}{k}$.

Do they form a basis? Let's see it for m/m^2 :

$f \in m^2$ is of the form $\sum_i f_i g_i$ where $f_i(0) = 0 = g_i(0)$

$\Rightarrow \partial_{x_i} f|_0 = 0 \quad \forall x_i$.

Then: assume $\sum a_i x_i \in m^2 \Rightarrow \partial_{x_j} (\sum a_i x_i) = a_j = 0 \quad \forall x_j$

$\Rightarrow a_i = 0 \quad \forall i \Rightarrow x_i$'s are l. indep. on $\mathbb{R}$.

(ii) $R = \mathbb{R}[\![x,y]\!]/(xy)$ this is local by point (6).

$\cup$

$m = (x,y)$ is the max ideal.

$m^2 = (x^2, \cancel{xy}, y^2) = (x^2, y^2)$

$m^k = (x^k, \cancel{x^{k-1}y}, \dots, \cancel{xy^{k-1}}, y^k) = (x^k, y^k)$

$m^k/m^{k+1} = \mathbb{R}x^k \oplus \mathbb{R}y^k \Rightarrow \dim_{\mathbb{R}}(m^k/m^{k+1}) = 2 \quad \forall k$

(iii) $R = \mathbb{R}[\![x,y]\!]/(x^2-y^3)$, $m = (x,y)$

$(x,y)^2 = (\underset{\overset{\shortparallel}{\cancel{y^3}}}{\cancel{x^2}}, xy, y^2) = (xy, y^2)$; $(x,y)^3 = (xy, y^2)(x,y) = (\underset{\overset{\shortparallel}{y^4}}{\cancel{x^2y}}, xy^2, y^3) = (xy^2, y^3)$

$\rightsquigarrow (x,y)^k = (xy^{k-1}, y^k)$, $(x,y)^{k+1} = (xy^k, y^{k+1})$

$(x,y)^k/(x,y)^{k+1} = \mathbb{R}xy^{k-1} \oplus \mathbb{R}y^k$, $\dim_{\mathbb{R}}(m^k/m^{k+1}) = 2$, $\forall k$.

② (a) $R = \mathbb{K}[\![x_1, \dots, x_n]\!]$ is a $\mathbb{K}$-algebra:

commutative

- $\left(\sum_{\nu \in \mathbb{N}^n} a_\nu \cdot x^\nu \right) + \left(\sum_{\nu \in \mathbb{N}^n} b_\nu \cdot x^\nu \right) = \sum_{\nu \in \mathbb{N}^n} (\underbrace{a_\nu + b_\nu}_{\in \mathbb{K}}) x^\nu \in R$ ✓

- $\left(\sum_{\nu \in \mathbb{N}^n} a_\nu \cdot x^\nu \right) \cdot \left(\sum_{\nu \in \mathbb{N}^n} b_\nu x^\nu \right) = \sum_{\nu \in \mathbb{N}^n} \Big(\underbrace{\sum_{\lambda + \mu = \nu} a_\lambda b_\mu}_{\in \mathbb{K}} \Big) x^\nu \in R$ ✓

commutative

- $\overset{\mathbb{K}}{\overset{\downarrow}{c}} \left(\sum_{\nu \in \mathbb{N}^\nu} a_\nu x^\nu \right) = \sum_{\nu \in \mathbb{N}^\nu} \underbrace{c\, a_\nu}_{\in \mathbb{K}} x^\nu \in R$

- $1 \in R$ is the unit ✓

$\Rightarrow$ R is a $\mathbb{K}$-algebra.

$\mathfrak{m} = \left\{ \sum_{\nu \in \mathbb{N}^n} a_\nu x^\nu \in R \;\middle|\; a_0 = 0 \right\}$ is max:

$$\begin{aligned} R &\xrightarrow{ev_0} \mathbb{K} \\ \sum a_\nu x^\nu &\longmapsto a_0 \end{aligned}$$

is surj with $\ker(ev_0) = \mathfrak{m}$

$\Rightarrow R/\mathfrak{m} \cong \mathbb{K}$ a field $\Rightarrow \mathfrak{m}$ is maximal.

(b) Taylor develop:

$$T : R_n = \begin{cases} \mathcal{O}_n \\ \mathcal{E}_n \end{cases} \longrightarrow \mathbb{K}[[x_1, \dots, x_n]] \;, \quad \mathbb{K} = \mathbb{R}, \mathbb{C}$$

$$f \longmapsto \sum_{\nu \in \mathbb{N}^n} \Big(\frac{1}{\nu!} \underbrace{D_0^\nu f}_{\frac{\partial^\nu}{\partial \underline{x}^\nu} f|_0} \Big) x^\nu$$

well-def on germs.

This is an $\mathbb{K}$-algebra homeo: it respects $+$, $c\cdot$ ✓

To check: $T(fg) = T(f)\,T(g)$

$$T(fg) = \sum_{\nu \in \mathbb{N}^n} \Big(\frac{1}{\nu!} D_0^\nu (fg) \Big) x^\nu$$

$$T(f)T(g) = \sum_{\nu \in \mathbb{N}^n} \Big(\sum_{\lambda + \mu = \nu} \frac{1}{\lambda!} D_0^\lambda f \cdot \frac{1}{\mu!} D_0^\mu g \Big) x^\nu$$

Goal: $\cancel{\nu!}\, \frac{1}{\cancel{\nu!}} D_0^\nu (fg) = \sum_{\lambda + \mu = \nu} \underbrace{\nu! \frac{1}{\lambda!\,\mu!}}_{= \binom{\nu}{\lambda}} D_0^\lambda f \cdot D_0^\mu g$ ← we will prove it without the evaluation in 0

By induction on $|\nu|$:

$$D^\nu(fg) = \sum_{\lambda+\mu=\nu} \binom{\nu}{\lambda} D^\lambda f\, D^\mu g$$

- $D^0(fg) = fg$ ✓
- $\nu \Rightarrow \nu + e_i$, $\quad e_i = (0,\dots,0,\overset{\swarrow i^{th}\text{ pos.}}{1},0,\dots,0) \in \mathbb{N}^n$:

$$D^{\nu+e_i}(fg) = D^{e_i}(D^\nu(fg)) = D^{e_i}\Big(\sum_{\lambda+\mu=\nu}\binom{\nu}{\lambda} D^\lambda f\, D^\mu g\Big) =$$

$$= \sum_{\lambda+\mu=\nu}\binom{\nu}{\lambda}\Big[(D^{\lambda+e_i} f)(D^\mu g) + (D^\lambda f)(D^{\mu+e_i} g)\Big] =$$

$$= \underbrace{\sum_{\mu\le\nu}\binom{\nu}{\mu} D^{\nu-\mu+e_i} f\, D^\mu g}_{*} + \underbrace{\sum_{\mu\le\nu}\binom{\nu}{\mu} D^{\nu-\mu} f\, D^{\mu+e_i} g}_{**} = $$

isolate the first term

$$* \overset{\swarrow}{=} \cancel{\binom{\nu}{0}}^{1}(D^{\nu+e_i} f) g + \sum_{0<\mu\le\nu}\binom{\nu}{\mu} D^{\nu-\mu+e_i} f\, D^\mu g$$

$$** \overset{\nwarrow \text{isolate last term}}{=} \sum_{\mu<\nu}\binom{\nu}{\mu} D^{\nu-\mu} f\, D^{\mu+e_i} g + \cancel{\binom{\nu}{\nu}}^{1} f\, D^{\nu+e_i} g$$

$$\overset{\mu'=\mu+e_i}{=} \sum_{0<\mu'\le\nu}\binom{\nu}{\mu'-e_i} D^{\nu-\mu'+e_i} f\, D^{\mu'} g$$

$$= (D^{\nu+e_i} f) g + \sum_{0 \lneq \mu \leq \nu} \underbrace{\left[\binom{\nu}{\mu} + \binom{\nu}{\mu - e_i}\right]}_{\binom{\nu+e_i}{\mu}} (D^{\nu-\mu+e_i} f)(D^{\mu} g) + f D^{\nu+e_i} g$$

$$= \sum_{0 \leq \mu \leq \nu+e_i} \binom{\nu+e_i}{\mu} (D^{\nu-\mu+e_i} f)(D^{\mu} g).$$