Singularity theory - Exercise class 7:

1) $f(x,y) = x^p + y^p \in R_2$

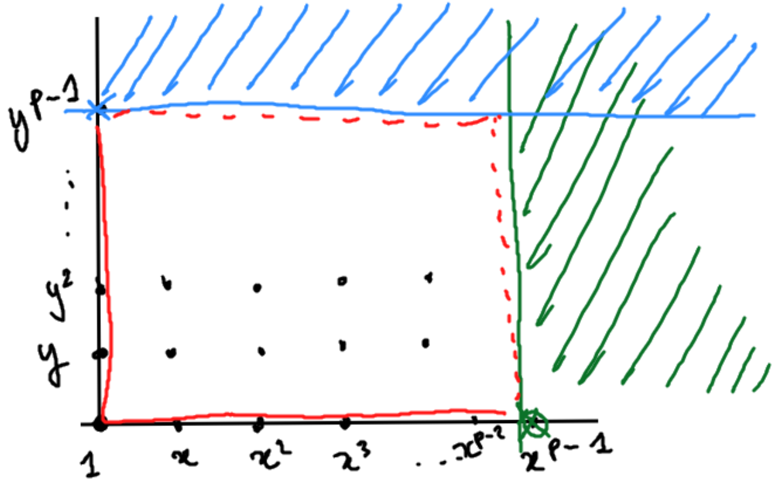
$g(x,y) = x^p + y^p + x^2y^2 \in R_2$, $p \geq 5$ $\Rightarrow f \not\sim g$.

"right equivalent"

We will show that f and g have different Milnor numbers.

- $Df = (px^{p-1}, py^{p-1}) \rightsquigarrow J_f = (x^{p-1}, y^{p-1})$

$\mu(f) = \dim_k \left(R_2/J_f \right) = \dim_k k[x,y]/(x^{p-1}, y^{p-1})$



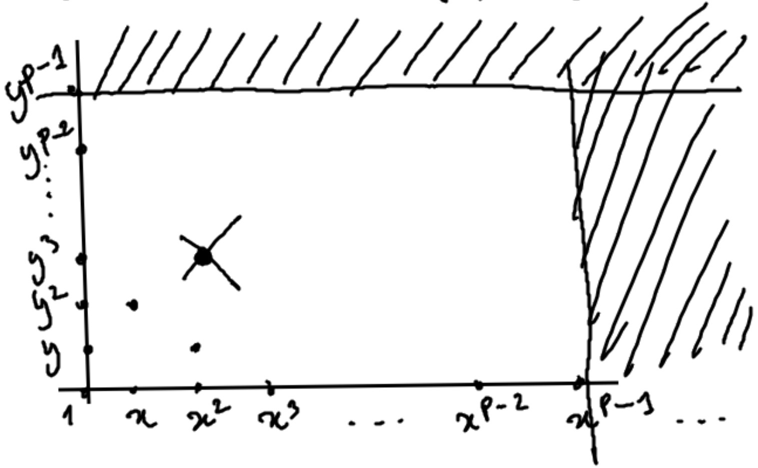
$\sum_{j,i \lneq p-1} a_{ij} x^i y^j \notin (x^{p-1}, y^{p-1})$

$\Rightarrow$ the elements in the square are a k-basis for R_2/J_f

$\Rightarrow \mu(f) = (p-1)^2$.

• $g(x,y) = x^p + y^p + x^2y^2$, $\quad p \geq 5$.

$Dg = (px^{p-1} + 2xy^2,\ py^{p-1} + 2x^2y) \rightsquigarrow J_g = (px^{\overset{\geq 4}{p-1}} + 2xy^2,\ py^{\overset{\geq 4}{p-1}} + 2x^2y)$



* $x^{p-1} \equiv -\frac{2}{p} xy^2 \quad \mod J_g$

$\Rightarrow$ x^{p-1} is linearly dep. on xy^2 (over k) in R_2/J_g

* same for y^{p-1}.

$\Rightarrow \mu(g) = \dim_k(R_2/J_g) \leq \mu(f)$

<u>Fact</u>: $y^{p+1} \in J_g$.

Then: $y^2(\underbrace{py^{p-1} + 2x^2y}_{\in J_g}) = py^{p+1} + 2x^2y^3 \in J_g$, but also $y^{p+1} \in J_g$

$\Rightarrow x^2y^3 \in J_g$, and x^2y^3 is not a multiple of $y^{\overbrace{p-1}^{\geq 4}}$ or $x^{\overbrace{p-1}^{\geq 4}}$

$\Rightarrow \mu(g) \lneq \mu(f)$.

4) $f = x^3 + xy^p \in R_2$, $p \geq 2$. Find the smallest $k \in \mathbb{N}$ s.th. $\mathfrak{m}^k \subset \mathfrak{m} J_f$.

[$\underset{\text{thm}}{\Rightarrow}$ the determinacy of f is k or $k-1$].

$Df = (3x^2 + y^p, pxy^{p-1}) \Rightarrow J_f = (3x^2+y^p, xy^{p-1})$. $\quad \mathfrak{m} = (x, y)$

$\mathfrak{m} J_f = (x,y)(3x^2+y^p, xy^{p-1}) = (3x^3 + \cancel{xy^p}, x^2y^{p-1}, 3x^2y + y^{p+1}, xy^p)$

$= (x^3, x^2y^{p-1}, 3x^2y + y^{p+1}, xy^p)$.

$\mathfrak{m} \supset \mathfrak{m}^2 \supset \dots \supset \mathfrak{m}^k \supset \dots$

$\mathfrak{m}^k \ni y^k$; $y^k \in \mathfrak{m} J_f \Leftrightarrow \exists f_1, f_2, f_3, f_4 \in R_2$ s.th.

$y^k = f_1 x^3 + f_2 x^2 y^{p-1} + f_3(3x^2y + y^{p+1}) + f_4 xy^p$

$\underset{\text{taking Taylor series}}{\rightsquigarrow}$ $y^k = T(f_1)x^3 + T(f_2)x^2y^{p-1} + T(f_3)(3x^2y + y^{p+1}) + T(f_4)xy^p$

$(T : R_2 \to k[[x,y]]$ homomorph. of algebras$)$

$\Rightarrow$ we need to get y^k as combinations of the generators of $\mathfrak{m}J_f$ also with coeff in $k[[x,y]]$.

$\Rightarrow$ in the combination, I need to include $3x^2y + y^{p+1}$

$\Rightarrow k \geq p+1$. Then $y^k = y^{p+1} \cdot y^a$, $a \geq 0$.

We said that we need to use $3x^2y + y^{p+1}$ $\rightsquigarrow$ we need to multiply it for a multiple of y^a $\rightsquigarrow$ $3x^2y \cdot y^a = 3x^2y^{a+1}$ $\leftarrow$ we need to get it from the other generators

$\Rightarrow$ we can get it only from $\begin{cases} x^2y^{p-1} \\ xy^p \end{cases}$ $\Rightarrow$ $a+1 \geq p-1$

$\Rightarrow a \geq p-2$.

In conclusion: $y^k \in mJ_f \Rightarrow k = p+1+a \geq p+1+p-2 = 2p-1$.

$\Rightarrow$ for $m^k \subseteq mJ_f$, we need $k \geq 2p-1$.

$mJ_f = (x^3, x^2y^{p-1}, 3x^2y + y^{p+1}, xy^p)$

Is $k = 2p-1$ fine? (is $m^{2p-1} \subseteq mJ_f$?).

$$m^{2p-1} = \{ \underset{\substack{\uparrow \\ mJ_f \\ \text{(just done)}}}{y^{2p-1}}, \underbrace{y^{2p-2}x}_{= xy^p \cdot y^{\overbrace{p-2}^{\geq 0}} \in mJ_f}, \underbrace{y^{2p-3}x^2}_{= x^2y^{p-1} \cdot \underbrace{y^{\overbrace{p-2}^{\geq 0}}}_{\in R_2} \in mJ_f}, \underbrace{\ldots\ldots}_{\text{multiples of } x^3 \in mJ_f} \} \subseteq mJ_f.$$

$(m = (x,y))$

2) $f(x,y,z) = x^3 + y^3 + z^3 + cxyz \in \mathbb{R}_3$, $c \in \mathbb{R}$, $c \neq 0, -1$.

$Df = (3x^2 + cyz, 3y^2 + cxz, 3z^2 + cxy) \rightsquigarrow J_f = (x^2 + cyz, y^2 + cxz, z^2 + cxy)$

$\boxed{c=-1}$: points on $\underbrace{\{x=y=z\}}_{\text{diagonal}}$ are critical $\Rightarrow$ no isolated sings.

$c \neq -1, 0$.

$(x^2 + cyz)y - (y^2 + cxz)cz + (z^2 + cxy)c^2x = (1+c^3)x^2y \in J_f$

But everything is symmetric in x, y, z $\rightsquigarrow$ $x^2z, xy^2, xz^2, y^2z, yz^2 \in J_f$.

Also: $(1+c^3)x^4 = (x^2 + cyz)(x^2 - cyz + c^3x^2) - (y^2 + cxz)c^3xy + (z^2 + cxy)c^2yz$

$\Rightarrow x^4, y^4, z^4 \in J_f$.

$\Longrightarrow \mathfrak{m}^4 \subseteq \mathfrak{m}^2 J_f \subseteq J_f \Longrightarrow f$ is 3-determined.

But f is a polynomial of deg 3 $\Rightarrow$ it is at least 3-determined $\Rightarrow$ 3 is the determinacy of f.

Furthermore:

$$R_3/J_f = k\langle 1, [x], [y], [z], [xy], [xz], [yz] \rangle$$

that's it: $[x^2] = [-cyz]$ in R_3/J_f

$\Rightarrow$ there's nothing of higher deg.

Are they lin. indep over k?

$$\cancel{a} + b\cancel{x} + \cancel{c}y + d\cancel{z} + \underbrace{exy + fyz + gxz}_{\underset{\text{mod } J_f}{=} \tilde{e}z^2 + \tilde{f}x^2 + \tilde{g}y^2} \in J_f$$

$\Rightarrow a = b = \ldots = g = 0$ $\Rightarrow$ $\mu(f) = \dim_k (R_3/J_f) = 7.$

4) $f = x^n$, $\quad F(x,t) = x^n + t\,x^{n+1} = f_t(x)$

(a) $H(x,t) : \underset{\substack{\cap \\ \mathbb{R}}}{I} \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ s.th $h_t(x) = H(x,t)$ is a coord change around 0

bounded intervall $\ni 0$

and s.th. $f \circ h_t = f_t$. $(*)$

$(*) \Leftrightarrow H(x,t)^n = x^n + t\,x^{n+1} = x^n(1 + t x)$

$\Rightarrow H(x,t) = \underbrace{x \cdot \sqrt[n]{1 + t x}}_{h_t(x)}$ is well def; $x \in I$ bounded intervall $\subseteq \mathbb{R}$

$I = (-a, b)$

$\rightsquigarrow$ for $\varepsilon \ll 1$, $t \in (-\varepsilon, \varepsilon)$, $1 + t x \geq 0$.

- $h_t(0) = 0 \quad \forall t$
- $h_t'(x) = \sqrt[n]{1 + t x} + \dfrac{xt}{n} \dfrac{\sqrt[n]{1 + t x}}{1 + t x} \Rightarrow h_t'(0) = 1 \neq 0$.

$\Rightarrow h_t$ is a coord change.

(b) $\xi(x,t)$ s.th: $\quad \xi(x,t)\cdot\partial_x F(x,t)+\partial_t F(x,t)=0 \quad [F(x,t)=x^n+tx^{n+1}]$.

$$\Leftrightarrow \quad \xi(x,t)\left(nx^{n-1}+t(n+1)x^n\right)=-x^{n+1}$$

$$\Leftrightarrow \quad \xi(x,t)=-\frac{x^{n+1}}{\underbrace{nx^{n-1}+t(n+1)x^n}_{x^{n-1}(n+t(n+1)x)}}=-\frac{x^2}{n+t(n+1)x}$$

smooth for $\varepsilon \ll 1$.

(c) $$\partial_t H(x,t)=\xi(H(x,t),t)=-\frac{H^2}{n+t(n+1)H}. \quad (**)$$

(d) H is a solution of $(**)$ $\Rightarrow \partial_t F(H(x,t),t)=0$.

we impose $(**)$

$$\partial_t(H^n+tH^{n+1})=nH^{n-1}\partial_t H+H^{n+1}+t(n+1)H^n\partial_t H=$$

$$=-\frac{nH^{n+1}}{n+t(n+1)H}+H^{n+1}-\frac{t(n+1)H^{n+2}}{n+(n+1)tH}=0 \quad \Leftrightarrow -nH^{n+1}-t(n+1)H^{n+2}+nH^{n+1}+(n+1)tH^{n+2}=0.$$

true!