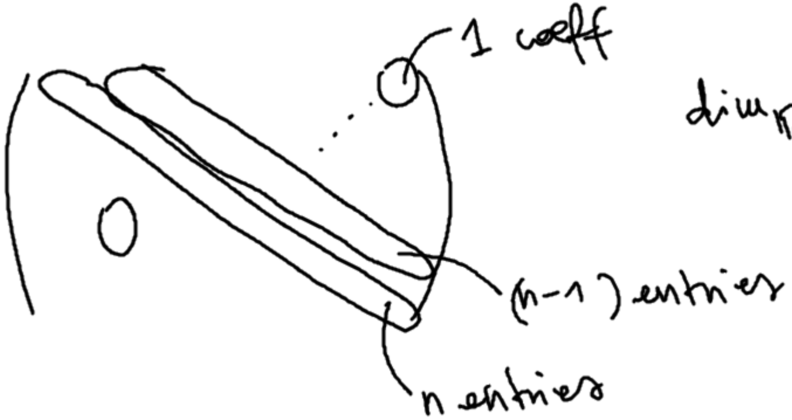
Ex class 3 - Singularity theory

Ex 1:

$$\begin{array}{c} \mathcal{T}(n,\mathbb{R}) \\ \cap \\ M_{n\times n}(\mathbb{R}) \end{array} = \left\{ \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix} \right\} , \qquad \begin{array}{c} \mathrm{Sym}(n,\mathbb{R}) \\ \cap \\ M_{n\times n}(\mathbb{R}) \end{array} = \left\{ \begin{pmatrix} \ddots & * \\ * & \ddots \end{pmatrix} \right\}$$

symmetric

(a) $\dim_{\mathbb{R}} \mathcal{T}(n,\mathbb{R}) = ?$ $\qquad \dim_{\mathbb{R}} \mathrm{Sym}(n,\mathbb{R}) = ?$



$$\dim_{\mathbb{R}} \mathcal{T}(n,\mathbb{R}) = n + (n-1) + (n-2) + \dots + 1$$
$$= \frac{(n+1)n}{2}$$

$\dim_{\mathbb{R}} \mathrm{Sym}(n,\mathbb{R})$ is the same: a symm. matrix is uniquely determined by its upper triangular part.

(b) $D_0 = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$, $d_i \neq 0 \quad \forall i \quad (\, d_i = \pm 1 \,)$

$$\alpha : \mathcal{T}(n,\mathbb{R}) \longrightarrow \mathrm{Sym}(n,\mathbb{R})$$
$$T \longmapsto T^{tr} \cdot D_0 \cdot T$$

$\left((T^{tr} \cdot D_0 \cdot T)^{tr} = T^{tr} \cdot D_0^{tr} \cdot (T^{tr})^{tr} = T^{tr} \cdot D_0 \cdot T \right)$

To show: α is a local diffeo

(i) α is smooth : $\mathcal{T}(n,\mathbb{R}) \subseteq M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$
$\mathrm{Sym}(n,\mathbb{R}) \subseteq M_{n\times n}(\mathbb{R})$

the differentiable structure on $\mathcal{T}(n,\mathbb{R}), \mathrm{Sym}(n,\mathbb{R})$ is the one induced by the embedding $\hookrightarrow \mathbb{R}^{n^2}$. $\leadsto$ α is smooth iff it is smooth as

$$\alpha : \mathcal{T}(n,\mathbb{R}) \longrightarrow \mathbb{R}^{n^2}$$
$$\cap\!|$$
$$\mathbb{R}^{n^2}$$

$\alpha : T \longmapsto T^{tr} D_0 T$ has a polynomial expression $\Rightarrow$ it is smooth.

(ii) Show that $(D\alpha)(Id) : T_{Id}\mathcal{T}(n,\mathbb{R}) \longrightarrow T_{\alpha(Id) = D_0} \mathrm{Sym}(n,\mathbb{R})$

is : $T \longmapsto T^{tr} D_0 + D_0 T$

$$\left[\mathcal{T}(n,\mathbb{R}) \cong \mathbb{R}^{\frac{n(n+1)}{2}} \cong \mathrm{Sym}(n,\mathbb{R}) \Rightarrow \begin{array}{l} T_{Id}\mathcal{T}(n,\mathbb{R}) \cong \mathcal{T}(n,\mathbb{R}) \\ T_{D_0}\mathrm{Sym}(n,\mathbb{R}) \cong \mathrm{Sym}(n,\mathbb{R}) \end{array} \right]$$

$$\rightsquigarrow (D\alpha)(Id) : \mathcal{T}(n,\mathbb{R}) \longrightarrow \mathrm{Sym}(n,\mathbb{R})$$

What is $D\alpha$? $\left(D\alpha(Id) \in M_{\underset{(i,j)}{n^2} \times \underset{(a,b)}{n^2}}(\mathbb{R}) \right)$

$$(D\alpha)_{(i,j),(a,b)} = \frac{\partial}{\partial t_{ij}} \left((T^t D_0 T)_{(a,b)} \right) =$$

$$\left(\underline{\text{Last time}}: \quad (T^{tr} D_0 T)_{(a,b)} = \sum_{k,l} t_{la} \underbrace{d_{lk}}_{\neq 0 \text{ only if } l=k} t_{kb} = \sum_k t_{ka} d_{kk} t_{kb} \right)$$

$$= \frac{\partial}{\partial t_{ij}} \left(\sum_k t_{ka} d_{kk} t_{kb} \right) = \sum_k \frac{\partial t_{ka}}{\partial t_{ij}} d_{kk} t_{kb} + \sum_k t_{ka} d_{kk} \frac{\partial t_{kb}}{\partial t_{ij}} =$$

$$= \delta_{aj} d_{ii} t_{ib} + t_{ia} d_{ii} \delta_{bj}$$

$$\left[\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \right]$$

$$\rightsquigarrow (D\alpha(Id))_{(i,j)(a,b)} = \delta_{aj} d_{ii} \delta_{ib} + \delta_{ia} d_{ii} \delta_{bj}$$

$$D\alpha(Id): \mathcal{T}(n,\mathbb{R}) \longrightarrow \mathcal{S}ym(n,\mathbb{R})$$

$$(D\alpha)(Id)(T)_{(i,j)} = \sum_{(a,b)} D\alpha(Id)_{(i,j)(a,b)}\, T(a,b) =$$

$$\left[\begin{array}{c} A \\ \begin{pmatrix} & & \\ & a_{ij} & \\ & & \end{pmatrix} \end{array} \begin{array}{c} v \\ \begin{pmatrix} \\ v_i \\ \end{pmatrix} \end{array} = \begin{pmatrix} \vdots \\ \sum_j a_{ij} v_j \\ \vdots \end{pmatrix} \leftarrow \text{pos. } i \right]$$

$$= \sum_{(a,b)} t_{ab} \left(d_{ii}\, \delta_{aj} \delta_{ib} + d_{ii}\, \delta_{bj}\, \delta_{ia} \right) =$$

$$= \underbrace{t_{ji}\, d_{ii}}_{(T^{tr} D_o)_{(i,j)}} + \underbrace{d_{ii}\, t_{ij}}_{(D_o T)_{(i,j)}} = \left(T^{tr} D_o + D_o T \right)_{(i,j)}$$

(iii) $(D\alpha)(\mathrm{Id})$ is injective $\left(\Rightarrow \text{it is surjective, because } \dim_{\mathbb{R}} \mathcal{T}(n,\mathbb{R}) = \dim_{\mathbb{R}} \mathrm{Sym}(n,\mathbb{R})\right)$

$$(D\alpha)(\mathrm{Id}) : \mathcal{T}(n,\mathbb{R}) \longrightarrow \mathrm{Sym}(n,\mathbb{R})$$
$$T \longmapsto T^{tr} \cdot D_0 + D_0 \cdot T$$

Rmk: $D_0 = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$ is s.th. $d_i \neq 0$ $\quad (d_i = \pm 1)$.

$$T = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ & t_{22} & \dots & t_{2n} \\ & 0 & \ddots & t_{nn} \end{pmatrix} \rightsquigarrow D_0 T = \begin{pmatrix} d_1 t_{11} & d_1 t_{12} & \dots & d_1 t_{1n} \\ & d_2 t_{22} & \dots & d_2 t_{2n} \\ & 0 & \ddots & d_n t_{nn} \end{pmatrix}$$

$$T^{tr} D_0 = (D_0 T)^{tr} \rightsquigarrow T^{tr} D_0 + D_0 T = \begin{pmatrix} 2 d_1 t_{11} & d_1 t_{12} & \dots & d_1 t_{1n} \\ & 2 d_2 t_{22} & & \\ * & & \ddots & 2 d_n t_{nn} \end{pmatrix}$$

$*$ the same (symm matrix)

Q: $T^{tr} D_0 + D_0 T = 0$? $\iff t_{ij} = 0 \ \forall i,j$ (using $d_i \neq 0$) $\iff T = 0$.

Ex 3:

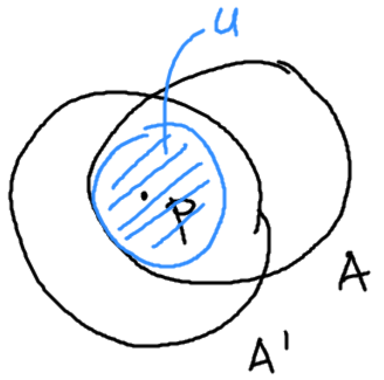
(a) $A, B \subseteq \mathbb{R}^n$ open $(p \in A, p \in B)$ $\rightsquigarrow$ (A,p), (B,p) germs of sets.

Check that $(A,p) \cap (B,p)$ is well-defined.

$$\begin{gathered}(A,p)\cap(B,p)\\ \parallel \\ (A\cap B, p)\end{gathered}$$

(1) $A \cap B \ni p$ ✓

(2) $(A,p) \sim (A',p)$, $(B,p) \sim (B',p)$ $\overset{?}{\Longrightarrow}$ $(A\cap B, p) \sim (A' \cap B', p)$



$(A,p) \sim (A',p) \Leftrightarrow \exists\, U \ni p$ open s.th. $A \cap U = A' \cap U$

$(B,p) \sim (B',p) \Leftrightarrow \exists\, V \ni p$ open s.th. $B \cap V = B' \cap V$

Then: $\underbrace{U \cap V}_{W} \ni p$ open, and $\begin{matrix} A \cap W = A' \cap W \\ B \cap W = B' \cap W \end{matrix}$ $\Rightarrow$ $(A \cap B) \cap W = (A' \cap B') \cap W$

$\Rightarrow (A \cap B, p) \sim (A' \cap B', p)$ ✓

(b) $(A,p)\times(B,q) := (A\times B, (p,q))$ is well-defined?

$\uparrow$ open $\nearrow$

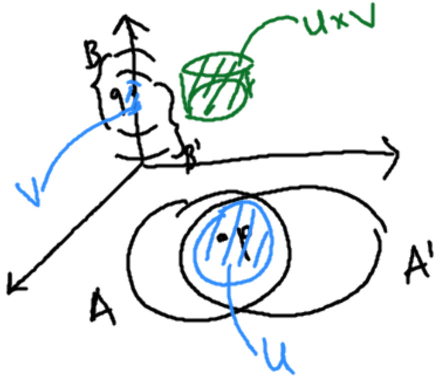
$A\subseteq\mathbb{R}^n,\ B\subseteq\mathbb{R}^m \rightsquigarrow A\times B\subseteq\mathbb{R}^{n+m}$

$(p,q)\in$

(1) $(p,q)\in A\times B$ ✓, $A\times B$ is open ✓
(def of product topology).

(2) $(A,p)\sim(A',p) \Leftrightarrow \exists\, U\ni p$ open, s.th $A\cap U = A'\cap U$

$(B,q)\sim(B',q) \Leftrightarrow \exists\, V\ni q$ open, s.th. $B\cap V = B'\cap V$



$$(A\times B)\cap\underbrace{(U\times V)}_{\text{open}\ \ni (p,q)} = (A'\times B')\cap(U\times V)$$

$\Rightarrow (A\times B,(p,q)) \sim (A'\times B',(p,q))$ ✓

(c) (f,p) map germ, $f: \underset{\substack{\cup \\ p}}{A} \longrightarrow Y$ (A contains $U \ni p$ open)

$\rightsquigarrow$ germ of the domain of $f = (A, p)$

Well defined (as a germ)?

$(f,p) \sim (g,p) \iff \exists \underset{\substack{\cup \\ p}}{U}$ open s.th. $f|_{A \cap U} = g|_{B \cap U}$

$f: A \to Y$

$g: B \to Y$

In particular, $A \cap U = B \cap U \implies (A,p) \sim (B,p)$ (as germs).

Ex 2

$f(x,y) = x^2 + y^2 + x^2y + xy^2 \in \mathcal{E}_{\mathbb{R}^2,0}$.

$\Big\}$ find local diffeo around 0

$g(x,y) = x^2+y^2 \in \mathcal{E}_{\mathbb{R}^2,0}$

. $f(0,0) = 0$, $Df = (2x + 2xy + y^2, 2y + x^2 + 2xy) \rightsquigarrow (Df)(0) = 0$

$D^2 f = \begin{pmatrix} 2+2y & 2x+2y \\ 2x+2y & 2+2x \end{pmatrix}$, $D^2 f(0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$\rightsquigarrow$ Morse lemma! $\exists\, \varphi$ local diffeo around 0 s.th.
$(\varphi(0)=0)$
$f(\varphi(x,y)) = x^2+y^2$.

How to construct φ?

$$f(x,y) = x^2 + y^2 + x^2 y + x y^2 = x^2(\underbrace{1+y}) + y^2(\underbrace{1+x})$$

around 0, their $\sqrt{\ }$ is well def!

$\rightsquigarrow \varphi(x,y) = (x\sqrt{1+y}, y\sqrt{1+x})$ well def $B_r(0)$ $(r<1)$

$\varphi(0,0) = (0,0)$

$$B_r(0) \xrightarrow{\varphi} \mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2$$

$$(x,y) \longmapsto (x\sqrt{1+y}, y\sqrt{1+x}) \mapsto x^2(1+y) + y^2(1+x) = f(x,y)$$

$\Rightarrow g(\varphi(x,y)) = f(x,y)$!

Can we invert φ? Yes, $D\varphi(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(around 0)

$\rightsquigarrow \psi = \varphi^{-1}$.