Singularity theory - Ex sheet 9

3) $f \in m_{\mathcal{E}_n}^2 = m_n^2$, $\mu(f) = 6$, $k := \text{corank}(f) := n - rk(D^2 f(0)) \geq 0$

(a) Lemma 4.7: $f \in m_n^2 \Rightarrow \mu(f) \geq \frac{1}{2} k(k+1)$

In our case: $6 > \frac{1}{2} k(k+1) \Leftrightarrow k(k+1) < 12 \Leftrightarrow k = 0, 1, 2$

$k=0$? We know: $k=0 \overset{\text{def}}{\Leftrightarrow} f$ non-deg $\underset{\text{Lemma 3.9}}{\Leftrightarrow} \mu(f) = 1$, while $\mu(f) = 6$ by assumption.

$\Rightarrow k \in \{1, 2\}$.

(b) $k=1$:

Lemma 4.5: $f \in m_n^2$, $\mu(f) < \infty$, $k=1 \Rightarrow f$ is stably equiv. to $\pm x_1^{\mu(f)+1}$

$\Updownarrow$.

$f \underset{R}{\sim} \pm x_1^{\mu(f)+1} + \underbrace{g(x_2, \dots, x_n)}_{\text{non deg. quadratic form.}}$

In our case: $\mu(f) = 6$

$\Rightarrow f$ stably equiv to $\pm x^7$.

(c) $k=2$.

Splitting lemma (thm 4.2): $f \in m_n^2$, $\underbrace{\ }_{k=\operatorname{cork}(f)} \Rightarrow \exists g \in m_k^3$, $a_{k+1}, \dots, a_n \in \{1,-1\}$ s.th.

$$f \underset{R}{\sim} g + \sum_{i>k}^{n} a_i x_i^2 .$$

In our case: $f \underset{R}{\sim} g + \sum_{i>2}^{n} a_i x_i^2$, $\quad a_i = \pm 1$. Is g 5-determined?

$g \in m_2^3$

Lemma 4.4 $\Rightarrow \mu(f) = \mu(g) = 6$.

$g \in m_2^3 \Rightarrow J_g \subset m_2^2 \Rightarrow \mathcal{A} := m_2 J_g \subset m_2^3$.

$$\underbrace{\dim \mathcal{E}_2/\mathcal{A}}_{(*)} = \underbrace{\dim(\mathcal{E}_2/m_2^3)}_{=\dim_{\mathbb{R}} \mathbb{R}[x,y]_{\leq 2} = \binom{4}{2} = \frac{4!}{2!\,2!} = 6} + \dim(m_2^3/\mathcal{A})$$

$\Rightarrow \dim(m_2^3/\mathcal{A}) = 2$.

$$(*) = \underbrace{\dim(\mathcal{E}_2/m)}_{1} + \underbrace{\dim(m/mJ_f)}_{=\dim(m/J_f) + \dim(J_f/mJ_f)} = 8$$

$\dim(m/J_f) = \mu - 1 = 5$, $\quad \underbrace{\dim(J_f/mJ_f)}_{2}$

Similarly to Ex 8.3(b):
$f \in m^2 \Rightarrow J_f = mJ_f + \mathbb{R}D^1 f + \dots + \mathbb{R}D^n f$
(in our case, $n=2$)

Ex 8.2, in the proof of (a): $\dim(\underbrace{\mathcal{A}+m_2^3/\mathcal{A}}_{c_3}) = \dim(m_2^3/\mathcal{A}) = 2$

$\Rightarrow m_2^5 \subset \mathcal{A} = m_2 J_g$

$\Rightarrow m_2^6 \subset m_2^2 J_g \underset{\text{thm 3.6}}{\Rightarrow} g$ is 5-determined.

(d) $g \in m_2^3 \Rightarrow g = p + h$, p cubic form in x,y $\overset{\text{Lecture 11}}{\Longrightarrow} p \in \{0, x^3, x^2y, x^3-xy^2, x^3+y^3\}$
$h \in m_2^4$

- 0 is not possible: Ex 8.3 (b): $f \in m_n^k$, $\forall\, l \geq 1 \rightsquigarrow \mu(f) \geq \binom{n+k+l-1}{k+l-1} - n\binom{n+l}{l}$

  in our case: $g = \cancel{p} + h = h$, $k=4$, $l=1$ $(n=2)$
  $\quad\;\, 0$

  $\rightsquigarrow \mu(g) = \mu(h) \geq \binom{6}{4} - 2\underbrace{\binom{3}{1}}_{6} = 9 \Rightarrow \mu(g) \geq 9$, but we're assuming $\mu(g) = 6$.

  $\binom{6}{4} = \frac{6!}{4!\,2!} = 15$ $\qquad$ $\mu(g) = \mu(f)$

- $\underbrace{x^3-xy^2, x^3+y^3}_{p}$: $\det(p) = 3$ (do it!)

  $\Rightarrow \det(g) = 3$ ↯. $\Rightarrow$ we can only have $p = x^3, x^2y$

(e) $p = x^3 \rightsquigarrow g = x^3 + h,\ h \in m_2^4.$

$\uparrow$ 5-det $\underset{\text{Ex 2}}{\Longrightarrow}$ $g \underset{R}{\sim} \begin{cases} \pm(x^3 + y^4) \\ \pm(x^3 + y^5) \\ x^3 + xy^3 + (j) \in m_2^5 \\ x^3 \pm xy^4 + (j) \in m_2^5 \end{cases}$

✓ can happen

they have wrong μ or determ (ex 6.2 & 6.3)

(d) $p = x^2 y \underset{\text{Ex 1}}{\Longrightarrow} g \underset{R}{\sim} \begin{cases} x^2 y \pm y^4 \\ x^2 y \pm y^5 \end{cases}$ only this has the right μ.

(1) $k \geq 4$; $f \in x^2y + \mathfrak{m}_2^k$ is equiv. to $x^2y \pm y^l$, $l \geq k$.

$\uparrow$ finitely det

$$\rightsquigarrow \quad f(x,y) = x^2y + \tilde{a}y^k + \tilde{b}xy^{k-1} + x^2 \overbrace{q(x,y)}^{\text{hom. pol. of deg } k-2} + \underbrace{\text{rest}}_{\in \mathfrak{m}_2^{k+1}}$$

apply $\varphi : (x,y) \mapsto \left(x - \frac{b}{2}y^{k-2},\ y - q(x,y)\right)$ local diffeo around 0.

$$x^2y + ay^k + \underset{\mathfrak{m}_2^{k+1}}{\tilde{r}}$$

apply a diffeo around 0

$$x^2y + \mu y^k + r, \quad \mu \in \{0, 1, -1\}$$

when $\mu \neq 0$, $f \sim_R x^2y + \mu y^{k+1}$

* $\mu \neq 0$: $\det(x^2y + \mu y^k) = k \Rightarrow f \sim_R x^2y + \mu y^k$.

* $\mu = 0$: $f(x,y) \in x^2y + \mathfrak{m}_2^{k+1} \Rightarrow$ we repeat the procedure $\rightsquigarrow f(x,y) \sim_R x^2y + \mu y^{k+1} + \underset{\mathfrak{m}_2^{k+2}}{r}$

if η is again 0, we go on. f fin. det $\Rightarrow$ we stop (at some point, we end with $\eta \neq 0$).
otherwise $f \in (x^2 y) + m^\infty$ $\lightning$.
not finitely det

$$\Rightarrow f \underset{R}{\sim} x^2 y + \eta y^\ell, \quad \ell \geq k.$$

(2) $k \geq 4$. $f \in x^3 + m_2^k$ $\Rightarrow$ $f = x^3 + \underbrace{h}_{\text{hom pol. of deg } k} + \underbrace{j}_{\in m_2^{k+1}} = x^3 + ay^k + bxy^{k-1} + x^2 \overbrace{p(x,y)}^{\text{hom. pol of deg } k-2} + j$
finitely det.

$(x,y) \mapsto (x - \frac{1}{3}p(x,y), y)$
(local diffeo around 0)

$x^3 + ay^k + bxy^{k-1} + j$

$$\Rightarrow f \underset{R}{\sim} x^3 + ay^k + bxy^{k-1} + \underbrace{j}_{\in m_2^{k+1}}.$$

Q: which of these germs are equivalent? i.e., when does it happen that, $a, b, a', b' \in \mathbb{R}$, $\exists \varphi = (\varphi_1, \varphi_2)$ s.th.

$$(*) \quad \varphi_1^3 + a\varphi_2^k + b\varphi_1\varphi_2^{k-1} \in x^2 + a'y^k + b'xy^{k+1} + m_2^{k+1} \quad ?$$

$\varphi_1^3 + a\varphi_2^k + b\varphi_1\varphi_2^{k-1} \in x^3 + a'y^k + b'xy^{k-1} + m_2^{k+1}$

$a, b, a', b' \in \mathbb{R}$

, $\varphi = (\varphi_1, \varphi_2)$ change of coord

- φ_1: the terms of φ_1 of order $\geq k-2$ can be omitted: $(k \geq 4)$
  - $3(k-2) = 3k-6 \geq k+1 \iff 2k \geq 7 \iff k \geq \frac{7}{2}$ true for $k \geq 4$ ✓
  - $2k-3 \geq k+1$ when $k \geq 4$
    $(\iff k \geq 4)$; also: the linear term of φ_1 is only in x
- φ_2: we can omitt the non-lin. terms.

$\varphi = (\varphi_1, \varphi_2): 0 \mapsto 0 \Rightarrow \varphi_1$ and φ_2 have not constant term

$\Rightarrow$ (*) $\iff$ $(x+q)^3 + a(\gamma x + \delta y)^k + bx(\gamma x + \delta y)^{k-1} \in x^3 + a'y^k + b'xy^{k-1} + m_2^{k+1}$

$q =$ pol without linear term, of deg $\leq k-1$

$\gamma, \delta \in \mathbb{R}$, $\delta \neq 0$: $D\varphi_1(0) = (1,0) \Rightarrow \underbrace{\partial_y \varphi_2(0)}_{=\delta} \neq 0$.

Step 1: $a\delta^{k} = a'$ holds (follows from [illegible])

$\Rightarrow$ either $a = a' = 0$, or they are both $\neq 0$, with the same sing when k is even.

Step 2: $a \neq 0 \neq a'$: $\rightsquigarrow$ $f \underset{R}{\sim} \eta^{k-1}(x^3 + y^k)$, $\eta = \mathrm{sgn}(a)$.

Step 3: $a = 0 = a'$ $\longrightarrow$ $b \neq 0$ $\rightsquigarrow$ $f \underset{R}{\sim} x^3 + \eta^k x y^{k-1} + m_2^{k+1}$

$\searrow$ $b = 0$: restart, with m_n^{k+1}. (similarly to ex 1)

one can check: the process stops.