Lecture 9: Kashiwara's constructibility theorem

recall from last time: X cplx. space $\rightsquigarrow D^b_c(X)$ derived cat. of complexes of $\mathbb{C}$-vector spaces with constructible cohomology. If X is algebraic, then $D^b_c(X) \overset{\text{full}}{\subset} D^b(\mathbb{C}_{X^{an}})$ (constructible w.r.t. algebraic stratification). $F^\cdot \in D^b_c(X)$ is perverse iff $\dim \operatorname{supp} \mathcal{H}^j(F^\cdot) \leq -j$ and $\dim \operatorname{supp} \mathcal{H}^j(D_X F^\cdot) \leq -j$ where $D_X F^\cdot = R\mathcal{H}om_{\mathbb{C}_X}(F^\cdot, \omega_X^\cdot)$

<u>Theorem (Kashiwara, Bernstein)</u>: Let X algebraic, and take $M^\cdot \in D^b_h(\mathcal{D}_X)$. Then $DR^\cdot_{X^{an}}(M^{an})$ and $Sol^\cdot_{X^{an}}(M^{an})$ are elements of $D^b_c(X)$. If $M \in Mod_h(\mathcal{D}_X)$, then $DR^\cdot_{X^{an}}(M^{an})$, $Sol^\cdot_{X^{an}}(M^{an})[n] \in Perv(X)$ (abelian subcat. of $D^b_c(X)$).

<u>Elements of proof</u>: as in alg. case, one shows that $Sol_{X^{an}}(M^{an}) = DR^\cdot_{X^{an}}(\mathbb{D}^{an} M^{an})$, hence it suffices to show the statement for either $Sol_{X^{an}}(M^{an})$ or $DR_{X^{an}}(M^{an})$. We give a simplified proof due to Bernstein in the alg. case (i.e., where M is alg. $\mathcal{D}_X$-module)

<u>Notational simplification</u>: Write $DR^\cdot(\mathcal{M}^\cdot)$ for $DR_{X^{an}}(\mathcal{M}^{an})$

It suffices to show $DR^\cdot(\mathcal{M}) \in D^b_c(X)$ for $\mathcal{M} \in \mathrm{Mod}_{h}(\mathcal{D}_X)$. We know that $\exists\, U \subset X$ open with $\mathcal{M}_{|U}$ is integrable connection. Then

<u>Lemma</u>: Let $\mathcal{N}$ be an int. connection on U then $DR^\cdot \mathcal{N}$ is (up to shift in homological degrees) a local system on U (i.e. $\ker(\nabla : \mathcal{E}^{an} \to \mathcal{E}^{an} \otimes \Omega^{1,an}_{U^{an}})$ is a local system if $(\mathcal{E}, \nabla)$ is int. connection).

In particular, we see that for $\mathcal{M} \in \mathrm{Mod}_{h}(\mathcal{D}_X)$, $\exists\, U$ s.t. $DR^\cdot_U(\mathcal{M}_{|U}) \in D^b_c(U)$

now we argue by decreasing induction on $\mathrm{supp}(\mathcal{M})$ and show the following claim:

$\mathcal{M} \in \mathrm{Mod}_{h}(\mathcal{D}_X)$, $U \subset X$ such that $DR_U(\mathcal{M}_{|U}) \in D^b_c(U)$ then $\exists\, Y \subset X \setminus U$ open s.t. $DR_{U \cup Y}(\mathcal{M}_{|U \cup Y}) \in D^b(U \cup Y)$

Let Z be irreducible component of $X \setminus U$,

$\exists$ open subset (local coordinate neighb.) $V \subset X$

and étale map $f: V \to V' \overset{\text{Zariski open}}{\subset} \mathbb{A}^n$ s.t.: $V \cap (X \setminus U) \overset{\text{dense}}{\subset} Z$

and $V' \cap \mathbb{A}^{n-k} \overset{\text{dense}}{\subset} \mathbb{A}^{n-k} = \{0\} \times \mathbb{A}^{n-k} \subset \mathbb{A}^n$ (heuristically: Z is locally given by $x_1 = \ldots = x_k = 0$ as subset of X).

f étale: $DR_V(M_{|V}) \in D^b_c(V) \iff f_* DR_V(M_{|V}) \in D^b_c(V')$

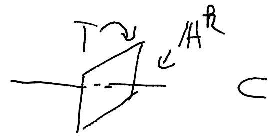
<u>Fact</u> (something to show): $f_* DR_V(M_{|V}) = DR_{V'}(\mathcal{H}^0 f_+ M_{|V})$

(this is not true for arbitrary maps, in gen. only $Rf_*(DR_V M_{|V}) \simeq DR_{V'}(f_+ M_{|V})$)

Hence we can assume from the beginning:

$X \subset \mathbb{A}^n$ open, $X \setminus U = \overbrace{X \cap \mathbb{A}^{n-k}}^{T :=} \simeq X \cap (\{0\} \times \mathbb{A}^{n-k}) \subset \mathbb{A}^n$

and $X \cap \mathbb{A}^{n-k} \subset \mathbb{A}^{n-k}$ dense. If we shrink X,

we can assume: $X \overset{\text{open}}{\subset} \mathbb{A}^k \times T$. (picture $X \subset$  $\subset \mathbb{A}^n$)

Now compactify $\mathbb{A}^k$ to $\mathbb{P}^k$, put $S := (\mathbb{P}^k \times T) \setminus X$, then

$\mathbb{P}^k \times T = S \cup \overbrace{T \cup U}^{X}$, S and T are closed in $\mathbb{P}^k \times T$

Write: $p: \mathbb{P}^k \times T \to T$ (projection), $j_X: X \hookrightarrow \mathbb{P}^k \times T$, $j_U: U \hookrightarrow \mathbb{P}^k \times T$ open embeddings

$\begin{array}{l} i_S : S \\ i_T : T \end{array} \hookrightarrow \mathbb{P}^k \times T$ closed embeddings. Let $N = j_{X,+} M$

in $D^b_h(\mathbb{P}^k \times T)$ and $K^\cdot := DR_{\mathbb{P}^k \times T}(N^\cdot) \in D^b(\mathbb{C}_{\mathbb{P}^k \times T}{}^{an})$.

We have adjunktion triangle (this time in $D^b_c(\mathbb{P}^k \times T)$):

$$(*) \quad j_{U!} j_U^{-1} K^\cdot \longrightarrow K^\cdot \longrightarrow i_{S!} i_S^{-1} K \oplus i_{T!} i_T^{-1} K^\cdot \xrightarrow{+1}$$

(in general, if $Y \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} X \setminus Y$ with i closed, and j open, then $\forall K \in D^b(X) = D^b(\mathbb{C}_X{}^{an})$ resp. in $D^b_c(X)$ $\exists$ dist. triangles: $j_! j^{-1} K \to K \to i_! i^{-1} K \xrightarrow{+1}$

& $i_* i^! K \to K \to j_* j^{-1} K \xrightarrow{+1}$)

apply Rp_* to $(*)$ we obtain: (notice $Rp_! = Rp_*$ as p proper)

$$\left[R(p\circ j_U)_! \underbrace{j_U^{-1} K}_{\substack{\shortparallel \\ \underbrace{j_U^{-1} DR_{\mathbb{P}^k \times T}(N)}_{\substack{\shortparallel \\ DR_U(N_{|U}) \in D^b_c(U)}}}} \to Rp_* K \to R(p\circ i_S)_! i_S^{-1} K \oplus R(p \circ i_T)_! i_T^{-1} K \xrightarrow{+1} \right]$$

$(**)$

induction assp.

$\Rightarrow$ (last time: any alg. map preserves $D^b_c(-)$)

$R(p\circ j_U)_! j_U^{-1} K \in D^b_c(T)$

last time was stated: $Rp_* K = Rp_* DR_{\mathbb{P}^k \times T}(N)$

$$\underset{p\text{ proj}}{\simeq} DR_T(p_+ N), \text{ but } p_+ N \in D^b_h(\mathcal{D}_T)$$

$\Rightarrow \exists\, Y \subset T$ open: $\mathrm{char}\left((p_+ N)_{|Y}\right) = \bigcup_k \mathrm{char}(\mathcal{H}^k p_+ N_{|Y}) \subset T^*_Y Y$

hence $Rp_* K_{|Y^{an}} \in D^b_c(Y) \xRightarrow{(**)} \left[R(p \circ i_T)_! \, i_T^{-1} K^\cdot\right]_{|Y^{an}} \in D^b_c(Y)$

but $p \circ i_T = id_T$, hence $i_T^{-1} K^\cdot = i_T^{-1} DR(j_{X,+} M)$.

but $i_T^{-1} DR(j_{X,+} M) = i^{-1} DR(M)$, where $i : T \hookrightarrow X$.

(recall that $T := X \setminus \mathcal{U}$). Hence we get

$\left[i^{-1} DR_X(M)\right]_{|Y^{an}} \in D^b_c(Y) \rightsquigarrow DR_{\mathcal{U} \cup Y}(M_{|\mathcal{U} \cup Y}) \in D^b_c(\mathcal{U} \cup Y)$ $\square$

Now let X cplx. mf. M analytic $\mathcal{D}_X$-modul

Sketch of pf of statement: $M \in Mod_h(\mathcal{D}_X) \Rightarrow Sol_X(M)[n] \in Perv(X)$: it suffices to show:

$M \in Mod_h(\mathcal{D}_X) \Longrightarrow \dim \mathrm{supp}\, \mathcal{H}^j(Sol^\cdot_X(M))[n] \leq -j$

since $\mathbb{D} M$ is also in $Mod_h(\mathcal{D}_X)$.

Some preliminaries:

1.) non-characteristic inverse images: let $X \overset{f}{\subset} Y$ be an embedding of a smooth subvar., let $M \in \mathrm{Mod}_c(\mathcal{D}_Y)$, then we say that X (or f) is non-characteristic w.r.t. M if

$$\omega^{-1}(\mathrm{char}(M)) \cap \underbrace{(T^*f)^{-1}(T^*_X X)}_{T^*_X Y} \overset{!}{\subset} X \times_Y T^*_Y Y$$

(then $T^*f_{|\omega^{-1}(\mathrm{char}\, M)} : \omega^{-1}(\mathrm{char}\, M) \to T^*X$ is finite)

recall that we have cotangent map:

$$T^*Y \xleftarrow{\ \omega\ } X \times_Y T^*Y \xrightarrow{T^*f} T^*X$$

<u>ex.</u>: $f: X \hookrightarrow Y$ embedding of smooth hypersurface.

$\rightsquigarrow T^*_X Y$ is line bundle on X. Let $P \in \mathcal{D}_Y$, $\mathrm{ord}(P) = m$

$M = \mathcal{D}_Y / \mathcal{D}_Y \cdot P$. $\mathrm{char}(M) = V(\sigma(P))$ (notice: M not hol. if $\dim Y > 1$)

hence f non-charact. w.r.t $M \iff \sigma(P)(x, \xi) \neq 0$

$\forall x \in X, \xi \in \mathrm{ann}(T_x X) \setminus \{0\} \subset T^*_x Y$, i.e. $(x, \xi) \in T^*_X Y \setminus 0$-section

$H^0 f^+ M$ is $\mathcal{D}_X$-loc. free of rk m (exercise)

<u>Theorem (pf omitted):</u> $f: X \hookrightarrow Y$ non-char

w.r.t $M \in \mathrm{Mod}_c(\mathcal{D}_Y) \Rightarrow$

1.) $H^j(f^+M) = 0 \quad \forall j \neq 0$

2.) $H^0(f^+M) \in \mathrm{Mod}_c(\mathcal{D}_X)$

3.) $\mathrm{char}\, H^0 f^+ M \subset (T^*f)(\omega^{-1}(\mathrm{char}\, M))$

2.) Structure of char(M) for holonomic M:

Let Y be smooth, $X \subset Y$ be arbitrary reduced subvariety. Then define:

$$T^*_X Y := \overline{T^*_{X_{reg}} Y} \subset T^* Y$$ conormal <u>space</u> of X in Y.

<u>Fact:</u> $T^*_X Y$ is singular Lagrangian subvariety (in part. $\dim T^*_X Y = \dim Y$ independent of $\dim X$!)

<u>Ex:</u> $X = V(x^2 - y^3) \subset \mathbb{C}^2 = Y \Rightarrow T^*_X Y \subset \mathbb{C}^4$ is so-called "open Whitney umbrella", surface with

isolated sing given by 4 equations projecting to ordinary Whitney umbrella in $\mathbb{C}^3$ (singular surface with line as sing. locus)

<u>Lemma:</u> Let X be smooth and $M \in \mathrm{Mod}_{q}(\mathcal{D}_X)$. Then $\exists$ stratification $(X_\alpha)_{\alpha\in A}$ of X^{an} s.t.

$$\mathrm{char}\, M^{an} = (\mathrm{char}\, M)^{an} \subset \bigcup_{\alpha\in A} T^*_{X_\alpha} X.$$

<u>idea:</u> For any conic lagrangian $\Lambda \subset T^*X$, let $\Lambda = \bigcup_{i\in I} \Lambda_i$ irred. components. Put $Z_i := \pi(\Lambda_i)$ and take stratification $X = \coprod X_\alpha$ s.t. Z_i is union of strata. Then $\Lambda \subset T^*_{X_\alpha} X$ $\square$

Proof of perversity of $K^\cdot := \mathrm{Sol}^\cdot(M)[n] := R\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{O}_X)$: recall that we have to show: $\forall j \geq 0$: $\dim \mathrm{supp}\, \mathcal{H}^j K^\cdot \leq -j$

Take stratification (X_α) with $\mathrm{char}\, M \subset \bigcup_{\alpha\in A} T^*_{X_\alpha} X$

write $i_{X_\alpha}: X_\alpha \hookrightarrow X$, $\forall \alpha \in A$. By first part of constructibility theorem, $i_{X_\alpha}^* K^\cdot$ has locally constant cohomology. Fix j and put $Z := \operatorname{supp} \mathcal{H}^j(K^\cdot) \Rightarrow Z$ union of connected components of X_α's. Let $p \in Z_{reg}$ and take $\alpha \in A$ with $p \in X_\alpha$ (then $\dim X_\alpha = \dim Z$). Chose germ (Y, z) such that intersection $Y \pitchfork X_\alpha$ at p is transversal (in part. $\dim Y + \dim Z \overset{!}{=} n = \dim X$).

<u>exercise:</u> can choose Y s.t. Y is non-charact. w.r.t. M (this follows from $\operatorname{char}(M) \subset \coprod T^*_{X_\alpha} X$)

<u>Th. (Cauchy-Kowalevski-Kashiwara):</u> We have $K^\cdot_{|Y} =$

$$\mathrm{Sol}^\cdot(M)_{|Y}[n] \simeq R\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{O}_X)_{|Y}[n]$$

$$\overset{!}{\simeq} R\mathcal{H}om_{\mathcal{D}_Y}(M_{|Y}, \mathcal{O}_Y)[n]$$

true for $M \in \mathrm{Mod}_c(\mathcal{D}_X)$ and general $Y \subset X$

here $M_{|Y}$ means $k^* M = H^0 k^+ M$, where $k: Y \hookrightarrow X$.

<u>idea:</u> essential case: $\operatorname{codim}_X Y = 1$, $X = \{x_1 = 0\}$, $M = D_X / P$, $\operatorname{ord}(P) = m$

$$k^{-1} \mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{O}_X) \longrightarrow \mathcal{H}om_{\mathcal{D}_Y}(k^+ M, \mathcal{O}_Y) \underset{\text{loc.}}{\overset{\sim}{=}} \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_Y^m, \mathcal{O}_Y) \simeq \mathcal{O}_Y^m$$

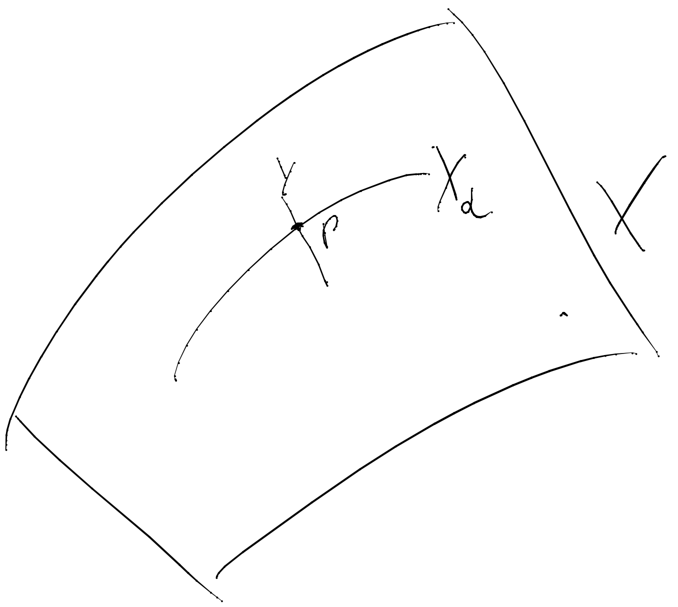
$$\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{O}_X) = \{u \in \mathcal{O}_X \mid Pu = 0\}$$

$$u \longmapsto (u_{|Y}, (\partial_{x_1} u)_{|Y}, \ldots, (\partial_{x_1}^{m-1} u)_{|Y})$$

is an isomorphism: fixing boundary values $(\partial_{x_1}^k u)_{|Y}$ for $k=0,\dots,m-1$ uniquely determines u.

(This is classical Cauchy-Kowalevski-Theorem)

geometric situation:



Since $p \in X_\alpha \subset Z = \mathrm{supp}\,\mathcal{H}^j(K^\cdot) = \mathrm{supp}\,\mathcal{H}^{n+j}\mathrm{Sol}^\cdot M$ we have $\mathcal{H}^j(K^\cdot)_p = \mathcal{E}xt^{n+j}_{\mathcal{D}_X}(M,\mathcal{O}_X)_p \neq 0$. Hence we also get $\left[\mathcal{E}xt^{n+j}_{\mathcal{D}_X}(M,\mathcal{O}_X)_{|Y}\right]_p \neq 0$ and thus

$\mathcal{E}xt^{n+j}_{\mathcal{D}_Y}(M_{|Y},\mathcal{O}_Y)_p \neq 0$. Since we have

$R\mathcal{H}om_{\mathcal{D}_Y}(M_{|Y},\mathcal{O}_Y) \simeq R\mathcal{H}om_{\mathcal{D}_Y}(M_{|Y},\mathcal{D}_Y) \overset{L}{\otimes}_{\mathcal{D}_Y} \mathcal{O}_Y$, and since $\mathcal{E}xt^r_{\mathcal{D}_Y}(\mathcal{M}_{|Y}, \mathcal{D}_Y) = 0$ $\forall r > \dim Y$, we conclude that: $n + j \leq \dim Y = \dim X - \dim Z$

$\iff n + j \leq n - \dim Z \iff j \leq -\dim Z$

$\iff \dim Z = \dim \operatorname{supp} \mathcal{H}^j Sol(\mathcal{M}[n]) \leq -j$

$\square$