Lecture 8: Analytic D-modules and constructible sheaves

Recall that for X smooth algebraic and $M \in D^b_{rh}(\mathcal{D}_X)$, we have seen that $\mathbb{H}^k(DR^\bullet M)$ and $\mathbb{H}^k(Sol^\bullet M) = \mathbb{H}^k(DR^\bullet(\mathbb{D}M))$ are finite-dim. $\mathbb{C}$-vector spaces. However, these groups are not always the right object to look at.

<u>example:</u> Let $X = \mathbb{A}^1$ and $M = \mathcal{D}_X / \mathcal{D}_X(\partial_t - \lambda)$

Then $\mathcal{H}^0 Sol(M) = 0$ but $\mathcal{H}^0 Sol_{X^{an}}(M^{an}) = \underline{\mathbb{C}}_{X^{an}} \cdot \varphi$

where $X^{an} = \overset{\text{cplx mf}}{\mathbb{C}}$, $M^{an} = M \otimes_{\mathcal{O}_X} \overset{\text{sheaf of hol. fcts.}}{\mathcal{O}_X^{an}}$, $\varphi = e^{\lambda t} \in \mathcal{O}_{X^{an}}$, $\notin \mathcal{O}_X$. Similarly,

$DR_X(M) = 0$ but $DR_{X^{an}}(M^{an}) \neq 0$

<u>Aim</u>: brief overview on analytic D-modules, statement and partial proof of Kashiwara's constructibility theorem.

Let X be smooth algebraic $/\mathbb{C}$ and X^{an} the associated cplx. mf., we have continuous map $\iota: X^{an} \to X$ of top. spaces and morphism $\iota^* \mathcal{O}_X \to \mathcal{O}_X^{an}$

of sheaves of rings. Fact: $\mathcal{O}_X^{an}$ is $i^{-1}\mathcal{O}_X$-flat. We have sheaf of rings $\mathcal{D}_{X^{an}}$ (locally $\mathcal{D}_{X^{an}} = \{\sum_\alpha c_\alpha \cdot \partial_X^\alpha \mid c_\alpha \in \mathcal{O}_{X^{an}}\}$) and $\mathcal{D}_{X^{an}} = \mathcal{O}_{X^{an}} \otimes_{\mathcal{O}_X} \mathcal{D}_X$ is $i^{-1}\mathcal{D}_X$-flat. We have functors

$$(-)^{an} : \mathrm{Mod}(\mathcal{D}_X) \longrightarrow \mathrm{Mod}(\mathcal{D}_{X^{an}});\ \mathcal{M} \mapsto \mathcal{D}_{X^{an}} \otimes_{i^{-1}\mathcal{D}_X} i^{-1}\mathcal{M}$$

extending to $D^b(\mathcal{D}_X) \to D^b(\mathcal{D}_{X^{an}})$

<u>Prop.:</u> 1.) $(-)^{an} : \mathrm{Mod}_c(\mathcal{D}_X) \to \mathrm{Mod}_c(\mathcal{D}_{X^{an}})$ resp. $D^b_c(\mathcal{D}_X) \to D^b_c(\mathcal{D}_{X^{an}})$

2.) $M^\cdot \in D^b_c(\mathcal{D}_X)$: $(\mathbb{D} M^\cdot)^{an} = \mathbb{D}(M^{\cdot\, an})$ duality for analytic $\mathcal{D}$-mod, similar def.

3.) $f : X \to Y$ morphism of alg. varieties, $M \in D^b_c(\mathcal{D}_Y)$, then

$$(f^+ M^\cdot)^{an} = (f^{an})^+((M^\cdot)^{an})$$

4.) f as before, $M \in D^b_c(\mathcal{D}_Y)$, then $\exists$ can. morphism $(f_+ M)^{an} \to (f^{an})_+ M^{an}$ in $D^b(\mathcal{D}_{Y^{an}})$. If f is proj. an $M \in D^b_c(\mathcal{D}_X)$, then this is isomorphism.

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rk: $f : X \to Y$ alg. then $f_+ : D^b_{rh}(\mathcal{D}_X) \to D^b_{rh}(\mathcal{D}_Y)$, however, it $f_+^{an} : D^b_{rh/c}(\mathcal{D}_X^{an}) \to D^b_{rh/c}(\mathcal{D}_Y^{an})$ only if f is proper

<u>Proof of 4.)</u>: we have morphism

$$(f^{an})^{-1}\mathcal{D}_{Y^{an}} \underset{(f^{an})^{-1}\iota_Y^{-1}\mathcal{D}_Y}{\otimes} \iota_X^{-1}\mathcal{D}_{Y\to X} \longrightarrow \mathcal{D}_{Y^{an}\leftarrow X^{an}}$$

(using $\iota_Y \circ f^{an} = f \circ \iota_X$).

$$(f_+M)^{an} = \mathcal{D}_{Y^{an}} \underset{\iota_Y^{-1}\mathcal{D}_Y}{\otimes} \underbrace{\iota_Y^{-1} Rf_*}_{\downarrow} \left(\mathcal{D}_{Y\leftarrow X} \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} M^\cdot\right)$$

exists for all $K^\cdot \in D^b(f^{-1}\mathcal{D}_Y)$

$$\longrightarrow \mathcal{D}_{Y^{an}} \underset{\iota_Y^{-1}\mathcal{D}_Y}{\otimes} \overbrace{R(f^{an})_*\, \iota_X^{-1}}\left(\mathcal{D}_{Y\leftarrow X} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} M^\cdot\right)$$

$$\longrightarrow R(f^{an})_*\left[\underbrace{(f^{an})^{-1}\mathcal{D}_{Y^{an}} \underset{(f^{an})^{-1}\iota_Y^{-1}\mathcal{D}_Y}{\otimes} \iota_X^{-1}\mathcal{D}_{Y\leftarrow X} \overset{\mathbb{L}}{\underset{\iota_X^{-1}\mathcal{D}_X}{\otimes}} \iota_X^{-1}M^\cdot}_{\to \mathcal{D}_{Y^{an}\leftarrow X^{an}}}\right]$$

$$\longrightarrow R(f^{an})_*\left[\mathcal{D}_{Y^{an}\leftarrow X^{an}} \overset{\mathbb{L}}{\underset{\iota_X^{-1}\mathcal{D}_X}{\otimes}} \iota_X^{-1}M\right]$$

$$= R(f^{an})_*\left[\mathcal{D}_{Y^{an}\leftarrow X^{an}} \overset{\mathbb{L}}{\underset{\mathcal{D}_{X^{an}}}{\otimes}} \underbrace{\mathcal{D}_{X^{an}} \overset{\mathbb{L}}{\underset{\iota_X^{-1}\mathcal{D}_X}{\otimes}} \iota^{-1}M}_{M^{an}}\right]$$

$$= (f^{an})_+(M^{an})$$

Show that this is iso if f is projective.

Essentially relevant case: $f = X = \mathbb{P}^n \times Y \to Y$ projection.

Then: $(f_+ M)^{an} = \mathcal{O}_{Y^{an}} \otimes_{i_Y^{-1}\mathcal{O}_Y} i_Y^{-1} \mathbb{R} f_* DR^\bullet_{X/Y}(M)$

$(f^{an})_+ M^{an} = \mathbb{R}(f^{an})_* \left(DR^\bullet_{X^{an}/Y^{an}} (M^{an}) \right)$

<u>Claim:</u> $\forall k$: $\exists$ quism in $D^b_{q.c.}(\mathcal{O}_{Y^{an}})$:

$$\mathcal{O}_{Y^{an}} \otimes_{i_Y^{-1}\mathcal{O}_Y} i_Y^{-1} R^\bullet f_* (\Omega^k_{X/Y} \otimes_{\mathcal{O}_X} M) \simeq R^\bullet (f^{an})_* (\Omega^k_{X^{an}/Y^{an}} \otimes_{\mathcal{O}_{X^{an}}} M^{an})$$

This gives Prop. 4.) by Čech-complex

Pf. of claim: we have $\mathcal{O}_{X^{an}} \otimes_{i_X^{-1}\mathcal{O}_X} \Omega^k_{X/Y} \otimes_{\mathcal{O}_X} M \simeq \Omega_{X^{an}/Y^{an}} \otimes_{\mathcal{O}_{X^{an}}} M^{an}$

hence to show: $\mathcal{O}_{Y^{an}} \otimes_{i_Y^{-1}\mathcal{O}_Y} i_Y^{-1} R f_* (\Omega^k_{X/Y} \otimes_{\mathcal{O}_X} M) \simeq R(f^{an})_* (\Omega^k_{X^{an}/Y^{an}} \otimes_{\mathcal{O}_{X^{an}}} M^{an})$

this is isomorphism by GAGA (by replacing M by $F_\bullet M$ and noticing: $\Omega^k_{X/Y} \otimes_{\mathcal{O}_X} F_e M$ is $\mathcal{O}_X$-coherent; $M = \bigcup F_e M$, Rf_* commutes with $\bigcup$

Constructible sheaves: Let X be an analytic space we have derived cat $D^b(\mathbb{C}_X)$ of complexes of $\mathbb{C}$-vector spaces. For analytic morph. $f: X \to Y$, we have functors Rf_*, f^{-1} (as usual, i.e. $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ and

$$(f^{-1}\mathcal{G})(V) = \text{sheaf asso to } \lim_{f(V)\subset U} \mathcal{G}(U)),$$

but also $Rf_!$ $((f_!\mathcal{F})(U) = \Gamma_c(f^{-1}(U), \mathcal{F}))$ and $f^!: D^b(\mathbb{C}_Y) \to D^b(\mathbb{C}_X)$: right adjoint of $Rf_!$

(i.e. $\forall F^\cdot \in D^b(\mathbb{C}_X)$, $\mathcal{G}^\cdot \in D^b(\mathbb{C}_Y)$: $\mathrm{Hom}_{D^b(\mathbb{C}_X)}(F^\cdot, f^!\mathcal{G}^\cdot) \simeq \mathrm{Hom}_{D^b(\mathbb{C}_Y)}(Rf_! F^\cdot, \mathcal{G}^\cdot)$).

Def: X analytic space, put $\omega_X := a^! \mathbb{C}_{\{pt\}} \in D^b(X)$, where $a: X \to \{pt\}$. ω_X is called dualizing complex of X. Fact: X cplx mf. $\Rightarrow \omega_X = \underline{\mathbb{C}}_X[2\dim_{\mathbb{C}} X]$

For $F^\cdot \in D^b(\mathbb{C}_X)$, define $D_X F^\cdot := R\mathcal{H}om_{\mathbb{C}_X}(F^\cdot, \omega_X)$ to be the Verdier dual of $F^\cdot$

<u>Stratifications:</u> X analytic space, let $X = \coprod_{\alpha \in A} X_\alpha$ be a locally finite partition, where:

1.) X_α is locally closed in X, i.e. the intersection of an open and a closed subset
2.) X_α is a complex manifold
3.) $\forall \alpha : \exists B_\alpha \subset A : \overline{X_\alpha} = \coprod_{\beta \in B_\alpha} X_\beta$

Then (X_α) is called stratification of X.

<u>ex.:</u> - $X = \mathbb{C} = \mathbb{C}^* \cup \{0\}$
- $X = \mathbb{C}^2 = (\mathbb{C}^2 \setminus C) \cup (C \setminus \{0\}) \cup \{0\}$, where $C \subset \mathbb{C}^2$ is curve with isolated singularity at 0, e.g. $C = V(x^2 - y^3)$

<u>Def.:</u> Let X be analytic space, and F a sheaf of $\mathbb{C}$-vector spaces. Then F is called constructible, if $\exists$ stratification $X = \coprod_\alpha X_\alpha$ s.t. $\forall \alpha$, $F_{|X_\alpha}$ is a local system, i.e. a locally constant sheaf.

recall: X top. space. $\exists$ equivalence of categories

$$\mathcal{L}oc(X) \longrightarrow \{ g : \pi_1(X) \to Gl_d(\mathbb{C}) \}$$

local systems of $\mathbb{C}$-v.sp. of $\dim = d$ — rank d complex representations of $\pi_1(X)$

Def.: $D^b_c(X)$ is full subcat. of $D^b(X)$ consisting of complexes with constructible cohomology.

example: let $X = \mathbb{C}$ and consider $P = x\partial_x - \alpha \in \mathcal{D}^{an}_X$, $\alpha \in \mathbb{C}$

$\mathcal{M} = \mathcal{D}_X / \mathcal{D}_X(P) \in \mathrm{Mod}_{c}(\mathcal{D}^{an}_X)$. Then $S := \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}^{an}_X)$ is constructible w.r.t. stratification $\mathbb{C} = \mathbb{C}^* \cup \{0\}$, we have $S_{|\mathbb{C}^*} = \mathbb{C}_{\mathbb{C}^*} \cdot x^\alpha$ (local system of rank 1, corresponds to representation $\pi_1(\mathbb{C}^*) \simeq \mathbb{Z}\gamma \to Gl_1(\mathbb{C}) = \mathbb{C}^*$ given by $\gamma \mapsto e^{2\pi i \alpha}$, where γ is counter clockwise loop around 0)

and $S_{|\{0\}} \simeq \begin{cases} \mathbb{C} & \text{if } \alpha = 0, 1, 2, \dots \\ 0 & \text{if } \alpha \in \mathbb{C} \setminus \mathbb{N}_0 \end{cases}$ since x^α extends to 0 in this case, i.e. $S = j_*(S_{|\mathbb{C}^*})$

Now suppose that X is algebraic $/\mathbb{C}$ and let X^{an} be the assoc. analytic space. We consider loc. finite partitions $X = \coprod_{\alpha \in A} X_\alpha$ where X_α are locally closed smooth subvarieties with $\overline{X_\alpha} = \coprod_{\beta \in B_\alpha} X_\beta$. This yields stratification of X^{an} by $X^{an} = \coprod_{\alpha \in A} X_\alpha^{an}$. $F \in Mod(\mathbb{C}_X)$ is called constructible if $\exists\ X = \coprod_{\alpha \in A} X_\alpha$ s.t. $F_{|X_\alpha^{an}} \in Loc(X_\alpha^{an})$.

Let $D_c^b(X)$ be full subcat of $D^b(X^{an})$ of complexes with constructible cohomology.

<u>Theorem (without proof)</u>: - X alg. variety, then $\omega_X \in D_c^b(X)$.

$D_X : D_c^b(X) \circlearrowleft$ & $D_X^2 \simeq Id$

- $f: X \to Y \leadsto f^{-1}, f^! : D_c^b(Y) \to D_c^b(X)$ & $Rf_*, Rf_!$: $D_c^b(X) \to D_c^b(Y)$ (properness is not needed if $f: X \to Y$ is algebraic)

- $Rf_! = D_Y \circ Rf_* \circ D_X$

- let X, Y smooth (and $f : X \to Y$ alg.)

$M \in D^b_c(\mathcal{D}_X)$, then $\exists$ can. morphism
(c: coherent)

$$DR_{Y^{an}}\left((f_+ M)^{an}\right) \to Rf_* DR_{X^{an}}(M^{an}) \text{ in } D^b(\mathbb{C}_{Y^{an}})$$

(later we will see: actually in $D^b_c(Y) \subset D^b(\mathbb{C}_{Y^{an}})$)
(c: constructible)

if f is proj., then this is isomorphism

<u>Def.</u>: X cplx. space / alg. var., $F^\cdot \in D^b_c(X)$ is called perverse sheaf if

$\dim \operatorname{supp} \mathcal{H}^j(F^\cdot) \leq -j$, $\dim \operatorname{supp} \mathcal{H}^j(\mathbb{D}_X F^\cdot) \leq -j$ $\forall j \in \mathbb{Z}$

$\operatorname{Perv}(\mathbb{C}_X)$ full subcat. of $D^b_c(X)$ of perverse sheaves

<u>FACT</u>: $\operatorname{Perv}(\mathbb{C}_X)$ is abelian !!!

(Luis' talk in Stab-Seminar ...)

<u>Kashiwara's constructibility theorem:</u> Let $\mathcal{M} \in D^b_{rh}(\mathcal{D}_X)$, then $DR_{X^{an}}(\mathcal{M}^{an})$ (and $Sol_{X^{an}}(\mathcal{M}^{an})$) are in $D^b_c(X)$. Moreover, if $\mathcal{M} \in Mod_{rh}(\mathcal{D}_X)$, then both are perverse sheaves.

<u>remark:</u> - this holds more generally for any analytic holonomic $\mathcal{D}_X$-module on a complex manifold X, this is Kashiwara's original proof.

- here we discuss only the algebraic case, and we give a simplified proof due to Bernstein