Lecture 5: Direct & inverse images: examples & applications

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Some examples: Recall that for $X = Y \times Z \to Y$

M in $\mathrm{Mod}(\mathcal{D}_X)$, we have defined relative de Rham complex $DR_{X/Y}(M) := (\ldots \to \Omega^k_{X/Y} \otimes_{\mathcal{O}_X} M \to \ldots)$

special case: $Y = \{pt\}$: the de Rham complex of M:

$$DR(M) : (\ldots \to \Omega^k_X \otimes_{\mathcal{O}_X} M \to \Omega^{k+1}_X \otimes_{\mathcal{O}_X} M \quad)$$

$$\omega \otimes m \mapsto d\omega + \sum dx_i \wedge \omega \otimes \partial_{x_i} m$$

in particular: $a : X \to \{pt\}$ projection, then

$$a_+ M \simeq DR(M) \in D^b(\mathcal{D}_{\{pt\}}) = D^b(\mathbb{C}\text{-v.sp.})$$

ex.: $M = \mathcal{O}_X \rightsquigarrow DR(M) = DR(\mathcal{O}_X) = DR_X$

(algebraic) de Rham complex of X

X smooth Grothendieck comparison: $H^i DR_X = H^i(X^{an}, \mathbb{C})$

Let $f \in \mathbb{C}[x_1, \dots, x_n]$, seen as $f : \mathbb{A}^n \to \mathbb{A}^1_t$.

<u>Def.</u>: The Gauß-Manin cohomology of f is

$$H^{i+n}\left(f_+ \mathcal{O}_{\mathbb{A}^n}\right) \in \mathrm{Mod}\left(\mathcal{D}_{\mathbb{A}^1}\right).$$

write: $f = p \circ i_f$, where $i_f : \mathbb{A}^n \hookrightarrow \mathbb{A}^n \times \mathbb{A}^1_t$, $\underline{x} \mapsto (x, f(x))$

$p : \mathbb{A}^n \times \mathbb{A}^1_t \to \mathbb{A}^1_t$ projection.

<u>Then</u>: $i_{f,+} \mathcal{O}_{\mathbb{A}^n} \simeq \left(i_{f,*} \mathcal{O}_{\mathbb{A}^n}\right)[\partial_t]$ with action

$$g \in \mathcal{O}_{\mathbb{A}^n} \quad \begin{cases} \partial_t (g \otimes \partial_t^k) = g \otimes \partial_t^{k+1} \\ \partial_{x_i} (g \otimes \partial_t^k) = \frac{\partial g}{\partial x_i} \otimes \partial_t^k - g \cdot \frac{\partial f}{\partial x_i} \otimes \partial_t^{k+1} \\ x_i (g \otimes \partial_t^k) = x_i g \otimes \partial_t^{k+1} \\ t (g \otimes \partial_t^k) = g \cdot f \otimes \partial_t^k - k g \otimes \partial_t^{k-1} \end{cases}$$

notice also: $i_{f,+} \mathcal{O}_{\mathbb{A}^n} \simeq \mathcal{D}_{\mathbb{A}^n \times \mathbb{A}^1_t} \Big/ \left(t - f, \left(\partial_{x_i} + \frac{\partial f}{\partial x_i} \cdot \partial_t\right)_{i=1..n} \right)$

Now consider $p_+ i_{f,+} \mathcal{O}_{\mathbb{A}^n} \simeq DR^{\bullet+n}_{\mathbb{A}^n \times \mathbb{A}^1_t / \mathbb{A}^1_t}\left(\left(i_{f,*}\mathcal{O}_{\mathbb{A}^n}\right)[\partial_t]\right)$

$$= \left(\ldots \to \pi^* \Omega^{k+n}_{\mathbb{A}^n}[\partial_t] \xrightarrow{\nabla} \pi^* \Omega^{k+1+n}_{\mathbb{A}^n}[\partial_t] \to \ldots \right)$$

where $\pi : \mathbb{A}^n \times \mathbb{A}^1_t \to \mathbb{A}^n$ and where $\nabla(\omega \otimes \partial_t^{\ell}) \underset{\text{exercise!}}{=}$

$$d\omega - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge \omega \otimes \partial_t^{\ell+1} = d\omega - df \wedge \omega \otimes \partial_t^{\ell+1}$$

since $p : \mathbb{A}^n \times \mathbb{A}^1_t \to \mathbb{A}^1_t$ is affine, p_* is exact, moreover $p_*\left(i_{f,*} \Omega^{k+n}_{\mathbb{A}^n}\right) = f_* \Omega^{k+n}_{\mathbb{A}^n}$, hence

<u>Prop.</u>: The direct image complex $f_+ \mathcal{O}_{\mathbb{A}^n} \in D^b(\mathcal{D}_{\mathbb{A}^1_t})$ is represented by $\left(\ldots \to f_* \Omega^{k+n}_{\mathbb{A}^n}[\partial_t] \longrightarrow f_* \Omega^{k+1+n}_{\mathbb{A}^n}[\partial_t] \to \ldots \right)$

and we call:

$$H^0 f_+ \mathcal{O}_{\mathbb{A}^n} = \frac{\Omega^n_{\mathbb{A}^n}[\partial_t]}{(d - df \wedge - \otimes \partial_t)\, \Omega^{n-1}_{\mathbb{A}^n}[\partial_t]}$$ the Gauß-Manin-system of f

<u>The Fourier transformation</u>: Let $M \in \mathrm{Mod}(D_{\mathbb{A}^1})$
$$D_{\mathbb{A}^1} = \mathbb{C}[t]\langle \partial_t \rangle,$$

and consider $\hat{M} := M$ as $\mathbb{C}$-v.sp., with action of $D_{\hat{\mathbb{A}}^1_\tau}$: $\forall m \in M$:
$$\tau \cdot m := -\partial_t m$$
$$\partial_\tau \cdot m := t \cdot m$$

$$\boxed{\begin{array}{l} \text{if } P(t,\partial_t) f(t) = 0 \\ \Rightarrow \hat{P} \int e^{t\cdot\tau} f(t)\,dt = 0 \\ \text{where } \hat{P} = P(\partial_\tau, -\tau) \end{array}}$$

$\hat{M}$ is called Fourier(-Laplace) transform of M

<u>cohomological reinterpretation</u>:

Consider diagram
$$\begin{tikzcd} & \mathbb{A}^1_t \times \hat{\mathbb{A}}^1_\tau \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{A}^1_t & & \mathbb{A}^1_\tau \end{tikzcd}$$

<u>Claim (exercise)</u>: $\hat{M} = H^0 q_+ \left(p^+ M \otimes_{\mathcal{O}_{\mathbb{A}^1_t \times \hat{\mathbb{A}}^1_\tau}} \mathcal{E}^{t\cdot\tau} \right)$

and all $H^i($ — " — $) = 0 \ \forall i \neq 0$

here: $\mathcal{E}^{t\cdot\tau} \simeq \mathcal{O}_{\mathbb{A}^1_t \times \hat{\mathbb{A}}^1_\tau}$ as $\mathcal{O}$-modules with differential: $\nabla := d + d(t\cdot\tau)$

rk: M and $\hat{M}$ are in general quite different:

ex: $M = \mathbb{C}[t] \; (= \mathcal{O}_{\mathbb{A}^1}) = D_{\mathbb{A}^1_t} / (\partial_t)$

$\Rightarrow \hat{M} = \mathbb{C}[\partial_\tau] = D_{\hat{\mathbb{A}}^1_\tau} / (\tau)$

notice: $\forall m \in \hat{M} : \exists k : \tau^k \cdot m = 0$

$\Rightarrow \operatorname{supp}(\hat{M}) = \{0\} \subset \hat{\mathbb{A}}^1_\tau$

Kashiwara's equivalence (without proof):

Theorem: Let $i : X \hookrightarrow Y$ be a closed embedding.
Put $\operatorname{Mod}^X_{q.c.}(\mathcal{D}_Y) := \{ M \in \operatorname{Mod}_{q.c.}(\mathcal{D}_Y) \mid \operatorname{supp}(M) \subset X \}$.

Then:

1.) $i_+ : \operatorname{Mod}(\mathcal{D}_X) \to \operatorname{Mod}(\mathcal{D}_Y)$ induces equivalence $\operatorname{Mod}_{q.c.}(\mathcal{D}_X) \simeq \operatorname{Mod}^X_{q.c.}(\mathcal{D}_Y)$.

and $\mathrm{Mod}_c(\mathcal{D}_X) \simeq \mathrm{Mod}_c^X(\mathcal{D}_Y)$, with quasi-inverse given by $\mathrm{Mod}_{qc}^X(\mathcal{D}_Y) \ni N \longmapsto H^0 i^+[\mathrm{codim}_Y X](N) \in \mathrm{Mod}_{q.c}(\mathcal{D}_X)$

2.) $\forall N \in \mathrm{Mod}_{q.c}^X(\mathcal{D}_Y) : H^j i^+[\mathrm{codim}_Y X](N) = 0 \;\forall j \neq 0$

<u>example:</u> Let $X = \{0\} \overset{i}{\hookrightarrow} Y = \mathbb{A}^{n+1}$, then

$$\begin{array}{c} i_+ : \mathrm{Mod}_{q.c.}(\mathcal{D}_X) \longrightarrow \mathrm{Mod}_{q.c.}^X(\mathcal{D}_Y) \\ \wr\!\wr \\ \mathrm{Vect}/\mathbb{C} \end{array}$$

consider $\mathcal{O}_X = \mathbb{C}_{\{0\}}$, then $\mathcal{B}_{X|Y} := H^0 i_+ \mathcal{O}_X = \mathcal{D}_Y / \mathcal{D}_Y(x_0, \dots, x_n)$

$\simeq \mathbb{C}[\partial_{x_0}, \dots, \partial_{x_n}] \cdot \delta_{\{0\}}$ ← generator of $\mathcal{B}_{X|Y}$, Dirac distribution

solution to $\{x_i \cdot u = 0\}_{i=0\dots n}$

It follows that $\forall M \in \mathrm{Mod}_{q.c}^X(\mathcal{D}_Y) : M = \bigoplus \mathcal{B}_{X|Y}$

Application:

Theorem: $\mathbb{P}^n$ is D-affine.

(recall: 1.) $\Gamma(\mathbb{P}^n, -) : Mod(\mathcal{D}_{\mathbb{P}^n}) \to Mod(\Gamma(\mathbb{P}^n, \mathcal{D}_{\mathbb{P}^n}))$
is exact

2.) $\mathcal{M} \in Mod(\mathcal{D}_{\mathbb{P}^n}) : \Gamma(\mathbb{P}^n, \mathcal{M}) = 0 \Rightarrow \mathcal{M} = 0$)

(recall: X D-affine implies: $Mod(\mathcal{D}_X) \overset{\text{equivalence}}{\longleftrightarrow} Mod(\Gamma(X, \mathcal{D}_X))$)

Proof: Write $X = \mathbb{P}^n$, $Y := \mathbb{A}^{n-1} \setminus \{0\}$, $\pi : Y \twoheadrightarrow X$

let $\mathcal{M} \in Mod_{q.c}(\mathcal{D}_X)$, $\rightsquigarrow \mathbb{G}_m$ acts on $H := \Gamma(Y, \pi^* \mathcal{M})$

($\pi^*\mathcal{M} = \mathcal{O}_Y \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M}$) we have decomposition $H := \bigoplus_{\ell \in \mathbb{Z}} H_\ell$ where

$\mathbb{G}_m \ni z \mapsto z^\ell \cdot - \in Aut(H_\ell)$. Notice that $\Gamma(X, \mathcal{M}) \cong H_0$

(this all holds at the level of q.c. $\mathcal{O}_X$ resp. q.c. $\mathcal{O}_Y$-mod)

We have $x_i \cdot H_e \subset H_{e+1}$ and $\partial_{x_i} \cdot H_e \subset H_{e-1}$.

Put $E := \sum_{i=0}^{n} x_i \partial_{x_i} \in \Gamma(Y, Der(\mathcal{O}_Y))$: Euler vector field

define $E : \pi^* M \circlearrowleft$ by $E \otimes Id$ (E acts on $\mathcal{O}_X$), then

$$H_l = \{ u \in \Gamma(Y, \pi^* M) \mid Eu = l \cdot u \}.$$

Write $i : \{0\} \hookrightarrow \mathbb{A}^{n+1}$, $j : Y = \mathbb{A}^{n+1} \setminus \{0\} \hookrightarrow \mathbb{A}^{n+1}$.

Let exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ in $Mod_{q.c}(\mathcal{D}_X)$

be given, the $0 \to \pi^* M_1 \to \pi^* M_2 \to \pi^* M_3 \to 0$ ($\pi : Y \to X$ smooth)

We obtain: $0 \to j_* \pi^* M_1 \to j_* \pi^* M_2 \to j_* \pi^* M_3 \to R^1 j_* \pi^* M_1 \to \dots$ $(*)$

in $Mod_{q.c.}(\mathcal{D}_{\mathbb{A}^{n+1}})$. Obviously we have $supp(R^1 j_* \pi^* M_1) \subset \{0\}$.

Now use Kashiwara's equivalence: $\exists!\, N \in Mod_{q.c.}(\mathcal{D}_{\{0\}}) = Vect_{\mathbb{C}}$:

$$R^1 j_* \pi^* M_1 = H^1 j_+ \pi^* M_1 \simeq H^0 i_+ N \simeq \mathbb{C}[\partial_{x_0}, \partial_{x_1}, \dots, \partial_{x_n}] \underset{\mathbb{C}}{\otimes} N.$$

we have (excercise): $x_i \cdot (\partial_{x_j}^k \otimes u) = \delta_{ij} \cdot (-k) \cdot (\partial_{x_j}^{k-1} \otimes u)$

$\forall k \geq 0,\ u \in N$

hence: $E(\underline{\partial}^\alpha \otimes u) = -(|\alpha| + (n+1))(\underline{\partial}^\alpha \otimes u)$

Consequence: we have weight decomposition

$$R^1 j_* \pi^* \mathcal{M}_1 \simeq H^0 i_+ N \simeq \bigoplus_{k \in \mathbb{Z}_{<0}} (R^1 j_* \pi^* \mathcal{M}_1)_k$$

Now take $\Gamma(\mathbb{A}^{n+1}, -)$ (exact functor (!)) of sequence (*)

$$0 \to \Gamma(\mathbb{A}^{n+1}, j_* \pi^* \mathcal{M}_1) \to \dots \to \Gamma(\mathbb{A}^{n+1}, R^1 j_* \pi^* \mathcal{M}_1) \to \dots$$

by definition of j_*, we have $\Gamma(Y, \pi^* \mathcal{M}_i) = \Gamma(\mathbb{A}^{n+1}, j_* \pi^* \mathcal{M}_i)$ for $i = 1,2,3$. Take 0-eigenspaces in above sequence (notice that $\Gamma(Y, \pi^* \mathcal{M}_i)_0 = \Gamma(X, \mathcal{M}_i)$ $i = 1,2,3$) yields:

$$0 \to \Gamma(X, \mathcal{M}_1) \to \Gamma(X, \mathcal{M}_2) \to \Gamma(X, \mathcal{M}_3) \to 0$$

(since E has < 0 eigenvalues on $R^1 j_* \pi^* \mathcal{M}_1$).

This proves 1.) (i.e. exactness of $\Gamma(X, -)$ on $\mathrm{Mod}_{q.c.}(\mathcal{D}_X)$)

Now show 2.): Let $\mathcal{M} \in \mathrm{Mod}_{q.c.}(\mathcal{D}_X)$ with $\Gamma(X, \mathcal{M}) = 0$ be given. Suppose $\mathcal{M} \neq 0$.

We have: $\Gamma(Y, \pi^* \mathcal{M}) = \bigoplus_{k \in \mathbb{Z}} \Gamma(Y, \pi^* \mathcal{M})_k$ (eigenspace decomposition w.r.t. action of E).

$\pi : Y \to X$ is smooth ($\Rightarrow$ faithfully flat), hence $\pi^* \mathcal{M} \neq 0$. Hence $\exists k : \Gamma(Y, \pi^* \mathcal{M})_k \neq 0$.

<u>Case 1</u>: $k > 0$: take $\overset{\neq 0}{u} \in \Gamma(Y, \pi^* \mathcal{M})_k$. Suppose $\partial_{x_i} u = 0 \ \forall i$
$\Rightarrow E u = 0$ ↯ $(k > 0)$. $\Rightarrow \exists i : \partial_{x_i} u \neq 0$ in $\Gamma(Y, \pi^* \mathcal{M})_{k-1}$.

<u>ex.</u>: - $\exists j : \partial_{x_j} (\partial_{x_i} u) \neq 0$
- $\Gamma(Y, \pi^* \mathcal{M})_0 \neq 0$ ↯ since $\Gamma(Y, \pi^* \mathcal{M}) = \Gamma(X, \mathcal{M})$

<u>Case 2</u>: $k < 0$: take $\overset{\neq 0}{u} \in \Gamma(Y, \pi^* \mathcal{M})_k$. Supp. $x_i \cdot u = 0 \ \forall i$
$\Rightarrow \mathrm{supp}(u) = \{0\} = Z \subset \mathbb{A}^{n+1}$ ↯ hence $\exists i : x_i u \neq 0$ in $\Gamma(Y, \pi^* \mathcal{M})_{k+1}$

$\Rightarrow \Gamma(Y, \pi^* \mathcal{M})_0 = \Gamma(X, \mathcal{M}) = 0$ ↯ $\square$

Good filtrations, characteristic varieties and holonomic $\mathcal{D}$-modules

recall order filtration on $\mathcal{D}_X$ locally given by $F_k\mathcal{D}_X := \left\{ P = \sum_{|\underline{\alpha}| \leq k} a_{\underline{\alpha}}(\underline{x}) \partial_x^{\underline{\alpha}} \right\}$. We have $\mathrm{gr}^F_\bullet \mathcal{D}_X \simeq \pi_* \mathcal{O}_{T^*X}$, where $\pi : T^*X \to X$ is the cotangent bundle.

Let $\mathcal{M} \in \mathrm{Mod}_{q.c.}(\mathcal{D}_X)$. $(\mathcal{M}, F_\bullet)$ is called a filtered $\mathcal{D}_X$-module iff:

- $F_i\mathcal{M}$ is $\mathcal{O}_X$-quasi-coherent
- $F_i\mathcal{M} \subset F_{i+1}\mathcal{M}$
- $F_i\mathcal{M} = 0 \quad \forall i \ll 0$
- $\mathcal{M} = \bigcup_{i \in \mathbb{Z}} F_i\mathcal{M}$
- $(F_k\mathcal{D}_X) \cdot (F_i\mathcal{M}) \subset F_{k+i}\mathcal{M} \quad \forall i, k \in \mathbb{Z}$

(easy) exercise: If $(\mathcal{M}, F_\bullet)$ is filtered, then $\mathrm{gr}^F_\bullet \mathcal{M} := \bigoplus_{k\in\mathbb{Z}} \mathrm{gr}^F_k \mathcal{M}$ is a quasi-coherent module over $\mathrm{gr}^F_\bullet \mathcal{D}_X = \pi_* \mathcal{O}_{T^*X}$.

Next we discuss natural finiteness criterions for filtrations:

<u>Lemma</u>: Let $(\mathcal{M}, F_\bullet)$ be a filtered $\mathcal{D}_X$-module. Then the following cond. are equivalent:

1.) $\mathrm{gr}^F \mathcal{M}$ is $\pi_* \mathcal{O}_{T^*X}$-coherent

2.) $F_i \mathcal{M}$ is $\mathcal{O}_X$-coherent $\forall i$ and $\exists N \in \mathbb{N} : \forall i \geq N$:

$$\left(F_k \mathcal{D}_X\right) \cdot \left(F_i \mathcal{M}\right) = F_{k+i} \mathcal{M}$$

3.) Locally, we have $\phi : \mathcal{D}_X^m \twoheadrightarrow \mathcal{M}$ and $\exists p_1, \dots, p_m \in \mathbb{Z}$ s.t.:

$$\phi\left(F_{p-p_1} \mathcal{D}_X \oplus \dots \oplus F_{p-p_m} \mathcal{D}_X\right) = F_p \mathcal{M} \qquad \forall p \in \mathbb{Z}.$$

<u>Def</u>: A filtration $F_\bullet \mathcal{M}$ is called good if it satisfies these equivalent conditions.

<u>Proof of the lemma (sketch)</u>: Show 1.) $\Leftrightarrow$ 3.):

1.) $\Rightarrow$ 3.) locally on X, take $u_1, \ldots, u_m \in \mathcal{M}$ s.t. $u_i \in F_{p_i}\mathcal{M}$ (for $p_1 \ldots p_m \in \mathbb{Z}$) and such that $\{\sigma_{p_i}(u_i)\}_{i=1\ldots m}$ generates $gr.\mathcal{M}$ over $\pi_* \mathcal{O}_{T^*X}$ where $\sigma_{p_i}: F_{p_i}\mathcal{M} \twoheadrightarrow gr^F_{p_i}\mathcal{M}$ is the projection. exercise: $\forall p: F_p\mathcal{M} = \sum_{i:p_i \le p} (F_{p-p_i}\mathcal{D}_X)\cdot u_i$ (induction over p). Then set $\phi(e_i) = u_i$.

3.) $\Rightarrow$ 1.) clear.

2.) $\Leftarrow$ 3.) clear locally (i.e., $\exists N$ locally s.t. 2.) holds) but then N exists globally since X is quasi-compact (since it is Noetherian).

Corollary: 1.) $\mathcal{M} \in \mathrm{Mod}_c(\mathcal{D}_X)$ has globally defined good filtration.

2.) Let $F_\cdot, F_\cdot'$ be good filtrations on $\mathcal{M}$, then $\exists\, i_0 \gg 0 : \forall i \in \mathbb{Z} : F'_{i-i_0}\mathcal{M} \subset F_i\mathcal{M} \subset F'_{i+i_0}\mathcal{M}$.

Proof: 1.) It follows from last lemma that

$\mathcal{M} \in \mathrm{Mod}_c(\mathcal{D}_X) \Leftrightarrow \exists$ good filtration locally on X.

hence to show: $\mathcal{M}$ is $\mathcal{D}_X$-coherent $\Rightarrow \exists$ global good filt.:

Claim: $\exists\, \mathcal{F} \in \mathrm{Coh}(\mathcal{O}_X)$ s.t. $\mathcal{M} = \mathcal{D}_X \cdot \mathcal{F}$.

„Pf": Take affine cover $X = \bigcup_{i=1}^{N} U_i$ s.t. $\mathcal{M}_{|U_i} = \sum \mathcal{D}_{U_i} e_j$

Put $\mathcal{F}_i := \sum \mathcal{O}_{U_i} e_j \xRightarrow{\text{Fact}} \exists\, \hat{\mathcal{F}}_i \in \mathrm{Coh}(\mathcal{O}_X)$ extending $\mathcal{F}_i$

Put $\mathcal{F} := \sum_{i=1}^{N} \hat{\mathcal{F}}_i$. (End of pf of Claim). Then set $F_i\mathcal{M} := (F_i\mathcal{D}) \cdot \mathcal{F}$

2.) exercise $\square$ **Rk:** Statement 1.) is wrong for analytic $\mathcal{D}$-modules

<u>Characteristic varieties</u>: Let $\mathcal{M} \in \mathrm{Mod}_c(\mathcal{D}_X)$, $F_\bullet \mathcal{M}$ a good filtration on $\mathcal{M}$. Let $\pi : T^*X \to X$

<u>Def</u>: Consider the graded sheaf of modules

$$\widetilde{\mathrm{gr}_\bullet^F \mathcal{M}} := \mathcal{O}_{T^*X} \underset{\pi^{-1}\pi_*\mathcal{O}_{T^*X}}{\otimes} \pi^{-1} \mathrm{gr}_\bullet^F \mathcal{M}.$$

Then the characteristic variety of $\mathcal{M}$, denoted by $\mathrm{char}(\mathcal{M})$ is $\mathrm{supp}\left(\widetilde{\mathrm{gr}_\bullet^F \mathcal{M}}\right)$.

<u>Theorem</u>: 1.) $\mathrm{char}\,\mathcal{M}$ does not depend on $F_\bullet \mathcal{M}$

2.) $\forall\ 0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$ short exact sequence in $\mathrm{Mod}_c(\mathcal{D}_X)$, we have

$$\mathrm{char}(\mathcal{M}_2) = \mathrm{char}(\mathcal{M}_1) \cup \mathrm{char}(\mathcal{M}_3)$$

<u>Proof</u>: slightly tricky, uses point 2.) of Corollary on p. 14.

<u>Prop.</u>: Let $0 \neq \mathcal{M} \in \mathrm{Mod}_c(\mathcal{D}_X)$, then are equivalent:

1.) $\mathcal{M}$ is integrable connection
2.) $\mathcal{M}$ is $\mathcal{O}_X$-coherent
3.) $\mathrm{char}\, \mathcal{M} = T^*_X X \simeq$ (zero section of $T^*X \to X$)

Such $\mathcal{M}$ are called smooth $\mathcal{D}_X$-modules.

<u>Pf.</u>: 1.) $\Leftrightarrow$ 2.) has already been shown

<u>1.) $\Rightarrow$ 3.)</u>: Put $F_i \mathcal{M} := \begin{cases} 0 & \forall i < 0 \\ \mathcal{M} & \forall i \geq 0 \end{cases} \Rightarrow F_\cdot \mathcal{M}$ is good (recall that $\mathcal{M}$ is $\mathcal{O}_X$-locally free). It follows that $\mathrm{gr}^F \mathcal{M} \simeq \mathcal{M}$ as $\mathcal{O}_X$-module and that in of Θ_X in $\mathrm{gr}^F_\cdot \mathcal{D}_X$ lies in $\mathrm{Ann}_{\mathrm{gr}^F_\cdot \mathcal{D}_X}(\mathrm{gr}^F \mathcal{M})$ (since any $v \in \Theta_X \subset \mathcal{D}_X$ sends $\mathcal{M}$ to itself, i.e its class in $\mathrm{gr}^F_\cdot \mathcal{D}_X$ acts by 0). But the vanishing locus of $\mathrm{in}(\Theta_X)$ in $\mathrm{gr}^F_\cdot \mathcal{D}_X$ is exactly the zero section of $T^*X \to X$

$\Rightarrow \mathrm{char}(\mathcal{M}) \subset T^*_X X$, but in general: $\pi(\mathrm{char}(\mathcal{M})) \supset \mathrm{supp}(\mathcal{M}) \overset{\mathcal{M}\ \mathcal{O}_X\text{-loc. free}}{\Longrightarrow} \mathrm{char}(\mathcal{M}) = X$

3.) $\Rightarrow$ 2.) (the essential part)

The assertion „$\mathcal{M} \in \mathrm{Coh}(X)$" is local, so assume X is affine with local coord. system $(x_1, \dots, x_n)$, then $T^*X = X \times \mathbb{C}^n$. Suppose $\mathrm{char}(X) = T^*_X X$, then for any good filtration $F_\bullet \mathcal{M}$, we have

$$\sqrt{\mathrm{Ann}_{\underbrace{\mathcal{O}_X[\xi_1,\dots,\xi_n]}_{\Gamma(X, \mathrm{gr}^F_\bullet \mathcal{D}_X)}} (\mathrm{gr}^F_\bullet \mathcal{M})} = (\xi_1, \dots, \xi_n) =: I$$

$\xi_i = \sigma(\partial_{x_i}) \in \mathrm{gr}_1 \mathcal{D}_X$ symbol

$\Rightarrow I^N \subset \mathrm{Ann}_{\mathcal{O}_X[\xi]} (\mathrm{gr}^F_\bullet \mathcal{M})$ for some $N > 0$

hence $\partial_{\underline{x}}^{\underline{\alpha}} \cdot F_k \mathcal{M} \subset F_{k+N-1} \mathcal{M} \quad \forall \underline{\alpha} : |\underline{\alpha}| = N, \forall k$

$\forall j \gg 0$, we have $F_i \mathcal{D}_X \cdot F_j \mathcal{M} = F_{i+j} \mathcal{M}$ ($F_\bullet \mathcal{M}$ is good), hence

$$F_{N+j} \mathcal{M} = F_N \mathcal{D}_X \cdot F_j \mathcal{M} = \Big(\sum_{|\alpha| \le N} \mathcal{O}_X \partial_{\underline{x}}^{\underline{\alpha}} \Big) \cdot F_j \mathcal{M} \subset F_{j+N-1} \mathcal{M}$$

$\Rightarrow F_{j+1} \mathcal{M} = F_j \mathcal{M} = \mathcal{M} \ \forall j \gg 0$. $F_j \mathcal{M} \in \mathrm{Coh}(\mathcal{O}_X) \Rightarrow \mathcal{M} \in \mathrm{Coh}(\mathcal{O}_X)$ $\square$

example: let $I \subset \mathcal{D}_X$ be a left ideal and put $M := D/I \in \mathrm{Mod}_c(\mathcal{D}_X)$. Consider the filtration

$$F^{ord}_k M := \{ [P] \mid P \in F_k \mathcal{D}_X \}.$$

Consider also the filtration $F_k I := F_k \mathcal{D}_X \cap I$.

Then $gr^F_\cdot I = \{ \sigma(P) \mid P \in I \} \subset gr^F_\cdot \mathcal{D}_X$ and we have: $\mathrm{char}(M) = V(\sqrt{gr^F I}) \subset T^*X$.

BUT: If $I = (P_1, \dots, P_k)$ then we have

$$Ch(M) \subset V(\sigma(P_1), \dots, \sigma(P_k))$$

but not $\supset$ in general.

especially: $X = \mathbb{A}^1$, $I = (P)$, $P = \sum_{i=0}^{d} a_i(t) \cdot \partial_t^i$, $M = D_{\mathbb{A}^1}/I$,

then $\mathrm{char}(M) = V(gr^F_\cdot I) = V(\sigma(P)) = V(a_d(t) \cdot \xi^d)$

$= \{ (t, \xi) \in T^*\mathbb{A}^1 = \mathbb{A}^2 \mid a_d(t) = 0 \} \cup \{ \xi = 0 \}$ $T^*_{\mathbb{A}^1}\mathbb{A}^1$

Basic properties of characteristic varieties:

1.) $\mathrm{char}(\mathcal{M})$ is conical in the fibres of $\pi: T^*X \to X$, i.e. $\forall (x,\xi) \in \mathrm{char}(\mathcal{M}) \subset T^*X$, we have that $(x, t\cdot\xi) \in \mathrm{char}(\mathcal{M})$ $\forall t \in \mathbb{C}^*$

(clear since $\sqrt{\mathrm{Ann}_{\mathcal{O}_{T^*X}}\, gr^F_\bullet \mathcal{M}}$ is homogeneous in the fibre variables of $\pi: T^*X \to X$)

2.) recall that T^*X has natural structure of symplectic manifold, given in local coordinates $x_1, \dots, x_n$ (and dual coordinates $\xi_1, \dots, \xi_n$ in the fibres of $\pi: T^*X \to X$) by the 2-form $\omega = \sum d\xi_i \wedge dx_i \in \Omega^2_{T^*X}$.

We have Poisson bracket $\{\,,\}: \mathcal{O}_{T^*X} \times \mathcal{O}_{T^*X} \to \mathcal{O}_{T^*X}$ def. by $\{f,g\} = \sum_{i=1}^n \partial_{\xi_i} f \cdot \partial_{x_i} g - \partial_{x_i} g \cdot \partial_{\xi_i} f$. Then

<u>Th. (Gabber)</u>: Put $I := \sqrt{\mathrm{Ann}_{\mathcal{O}_{T^*X}}\, gr^F_\bullet \mathcal{M}}$, then I is involutiv, i.e.: $\{I, I\} \subset I$. This implies $\forall p \in \mathrm{char}(\mathcal{M})_{reg}: T_p \,\mathrm{char}\,\mathcal{M} \supset (T_p \,\mathrm{char}\,\mathcal{M})^{\perp,\omega}$. ($\mathrm{char}(\mathcal{M})$ co-isotropic)

in part. (can be shown independently): $\dim(\mathrm{char}(\mathcal{M})) \geq n$

<u>Definition</u>: Let $\mathcal{M} \in \mathrm{Mod}_c(\mathcal{D}_X)$. Then

$\mathcal{M}$ is called <u>holonomic</u> if $\dim(\mathrm{char}(\mathcal{M})) = n$

(easy excercise): $\Lambda \subset T^*X$ is co-isotropic + $\dim \Lambda = n$

$\Rightarrow \Lambda$ is isotropic, i.e. $\forall p \in \Lambda_{reg}: (T_p\Lambda) \subset (T_p\Lambda)^{\perp\omega}$, i.e.

$\mathcal{M}$ holonomic $\Rightarrow \mathrm{char}(\mathcal{M})$ is Lagrangian, that is:

1.) let $k: \mathrm{char}(\mathcal{M})_{reg} \hookrightarrow T^*X$, then $k^*\omega = 0$

2.) $\forall C \subset \mathrm{char}(\mathcal{M})$ irr. component: $\dim(C) = n$

simplest example: $\mathcal{M}$ integrable connection, then

$\mathrm{char}(\mathcal{M}) = X \subset T^*X$: this is Lagrangian $\Rightarrow \mathcal{M}$ hol.

other example: $\mathrm{supp}(\mathcal{M}) = \{pt\} \underset{\substack{\uparrow \\ \text{exercises} \\ \text{(more gen.)}}}{\Longrightarrow} \mathrm{char}(\mathcal{M}) = T^{\perp}_{pt}X$

$\subset T^*X$, this is also Lagrangian $\Rightarrow \mathcal{M}$ is hol

non-example: $\mathcal{M} = \mathcal{D}_X \Rightarrow \widetilde{\mathrm{ann}\, gr^F_\cdot \mathcal{M}} = (0) \Rightarrow \mathrm{char}\, \mathcal{M} = T^*X$ not Lag

$\Rightarrow \mathcal{D}_X$ is NOT holonomic

<u>Philosophy</u>: $\mathcal{M} = \mathcal{D}/\mathcal{I}$ hol. $\Leftrightarrow \dim \mathrm{char}\, \mathcal{M}$ min (small) $\Leftrightarrow \sqrt{\mathrm{Ann}\, \widetilde{gr^F_\cdot \mathcal{M}}}$ large

$\Leftrightarrow gr^F_\cdot \mathcal{I}$ large $\Leftrightarrow \mathcal{I}$ large $\Leftrightarrow Sol(\mathcal{M}) = \{f \mid Pf = 0\ \forall P \in \mathcal{I}\}$ small

<u>actually</u>: $\mathcal{M}$ hol. $\Rightarrow \dim_{\mathbb{C}}(Sol(\mathcal{M})) = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathcal{D}_X^{an}}(\mathcal{M}^{an}, \mathcal{O}_X^{an})) < \infty$