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Lecture 3: More about inverse & direct images

Reminder: - $\mathcal{D}_X$ sheaf of alg. differential operators

- left $\mathcal{D}_X$-modules: $\mathcal{O}_X$, $\mathcal{D}_X / I$, where I is left ideal (in particular: $\mathcal{D}_X$ is left $\mathcal{D}_X$-mod)

- right $\mathcal{D}_X$-modules: $\omega_X := \Omega^n_X$ $(\theta \mapsto (\omega \mapsto -Lie_\theta \omega))$ $\mathcal{D}_X / J$, if J is right ideal (in part.: $\mathcal{D}_X$ is right $\mathcal{D}_X$-module)

- tensor products / Hom-sheaves over $\mathcal{O}_X$ have left resp. right structures

- rk.: $\mathcal{D}_X^{opp}$ opposite (sheaf of) rings: $a^{opp}, b^{opp} \in \mathcal{D}_X^{opp} = \mathcal{D}_X$ as (sheaf of) $\mathbb{C}$-v.sp., $a^{opp} \cdot b^{opp} := b \cdot a$, then M' is right $\mathcal{D}_X$-module $\Longleftrightarrow$ $M \in Mod(\mathcal{D}_X^{opp})$: left modules over $\mathcal{D}_X^{opp}$

- $M \in Mod(\mathcal{D}_X) \rightsquigarrow \omega_X \otimes_{\mathcal{O}_X} M \in Mod(\mathcal{D}_X^{opp})$ asso. right module

- $M' \in Mod(\mathcal{D}_X^{opp}) \rightsquigarrow \mathcal{H}om_{\mathcal{O}_X}(\omega_X, M') = \Theta_X \otimes_{\mathcal{O}_X} M' \in Mod(\mathcal{D}_X)$ asso. left mod.

Recall from last time: $f: X \to Y$ morphism between smooth alg. varieties, $M \in Mod(\mathcal{D}_Y)$

left $\mathcal{D}_Y$-module $\Rightarrow f^*M := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}M = \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}M$

is left $\mathcal{D}_X$-mod., where $\mathcal{D}_{X\to Y} = f^*\mathcal{D}_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$

is $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bimodule, called transfer module.

if $X \overset{i}{\hookrightarrow} Y$ is closed embedding, locally given as $(x_1, \dots, x_r) \mapsto (x_1, \dots, x_r, 0, \dots, 0) =: (y_1, \dots, y_n)$, then

$$\mathcal{D}_{X\to Y} = \mathcal{D}_X[\partial_{y_{r+1}}, \dots, \partial_{y_n}] \simeq i^{-1}\mathcal{D}_Y / (y_{r+1}, \dots, y_n)\, i^{-1}\mathcal{D}_Y$$

given by: $h(\underline{x}) \otimes P(\underline{y}, \partial_{\underline{y}}) \mapsto [h(y_1, \dots, y_r) \cdot P(\underline{y}, \partial_{\underline{y}})]$

(compatible with left $\mathcal{D}_X$ and right $f^{-1}\mathcal{D}_Y$-structure)

Now suppose that $X \xrightarrow{f} Y$ is a projection, i.e. $X = Y \times T$, $f = pr_1$. Locally $f(x_1, \dots, x_n) = (x_1, \dots, x_r)$

$=: (y_1, \dots, y_r) \Longrightarrow \mathcal{D}_{X\to Y} = f^*\mathcal{D}_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y = \mathcal{O}_X[\partial_{y_1}, \dots, \partial_{y_r}]$

with left $\mathcal{D}_X$-action given by:

$$\partial_{x_i}\big(\underset{\mathcal{O}_X}{h} \otimes \underset{f^{-1}\mathcal{D}_Y}{P}\big) = \partial_{x_i}(h)\otimes P + h\otimes (Tf_*)(\partial_{x_i})\cdot P$$

$$= \begin{cases} \partial_{x_i}(h)\otimes P + h\otimes \partial_{y_i}\cdot P & \forall i\in\{1,\dots,r\} \\ \partial_{x_i}(h)\otimes P & \forall i\in\{r+1,\dots,n\} \quad (Tf_*(\partial_{x_i})=0 \text{ in this case}) \end{cases}$$

hence: $\underset{\partial_X \quad f^{-1}\mathcal{D}_Y}{\mathcal{D}_{X\to Y}} \simeq \dfrac{\mathcal{D}_X}{\mathcal{D}_X(\partial_{x_{r+1}},\dots,\partial_{x_n})}$

right $f^{-1}\mathcal{D}_Y$-structure since $P\in f^{-1}\mathcal{D}_Y$ commutes with $\partial_{x_{r+1}},\dots,\partial_{x_n}$

by $h(\underline{x})\otimes P(y,\partial_y) \longmapsto h(x)\cdot P(x_1,\dots,x_r,\partial_{x_1},\dots,\partial_{x_r})$

is isomorphism of left $\mathcal{D}_X$-modules and right $f^{-1}\mathcal{D}_Y$-modules

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Direct images, first version: basic idea:

use again the transfer module $\mathcal{D}_{X\to Y}$, but in "reversed" order, i.e.: let $f: X\to Y$, M a right $\mathcal{D}_X$-module

Then we can consider: $\mathcal{M} \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X\to Y}$

which is a right $f^{-1}\mathcal{D}_Y$-module, and

apply f_* yields functor.

$$f_*\left(- \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X\to Y}\right) : \mathrm{Mod}(\mathcal{D}_X^{opp}) \to \mathrm{Mod}(\mathcal{D}_Y^{opp})$$

<u>2 problems:</u> (a) How to define it on left modules

(b) $- \otimes \mathcal{D}_{X\to Y}$ is right exact, $f_*(-)$ is left exact $\rightsquigarrow$ direct image is neither right nor left exact

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Solution to (a): use left / right transformation recall:

$$\begin{tikzcd} \mathrm{Mod}(\mathcal{D}_X) \arrow[r, dashed] \arrow[d, "\omega_X \otimes_{\mathcal{O}_X} -"'] & \mathrm{Mod}(\mathcal{D}_Y) \\ \mathrm{Mod}(\mathcal{D}_X^{op}) \arrow[r, "f_*(- \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y})"] & \mathrm{Mod}(\mathcal{D}_Y^{op}) \arrow[u, "\mathcal{H}om_{\mathcal{O}_Y}(\omega_Y, -)"'] \end{tikzcd}$$

Hence for $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_X)$, direct image is $\mathcal{M} \mapsto \mathcal{H}om_{\mathcal{O}_Y}\left[\omega_Y, f_*\left((\omega_X \otimes_{\mathcal{O}_X} \mathcal{M}) \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}\right)\right]$

$$\simeq \underbrace{\omega_Y^\vee \otimes_{\mathcal{O}_Y}}_{} f_*\left((\omega_X \otimes_{\mathcal{O}_X} \mathcal{M}) \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}\right)$$

different description of the functor $\mathrm{Mod}(\mathcal{D}_Y^{op}) \to \mathrm{Mod}(\mathcal{D}_Y)$

exercise: $\exists$ isomorphism of right $f^{-1}\mathcal{D}_Y$-mod.

$$(\omega_X \otimes_{\mathcal{O}_X} \mathcal{M}) \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y} \simeq (\omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y}) \otimes_{\mathcal{D}_X} \mathcal{M}$$

by $(\omega \otimes m) \otimes P \longmapsto (\omega \otimes P) \otimes m$

(right action of $f^{-1}\mathcal{D}_Y$ on $(\omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y}) \otimes_{\mathcal{D}_X} \mathcal{M}$ by
$((\omega \otimes R) \otimes S)P := \omega \otimes RP \otimes S$)

hence: direct image can be rewritten as $M \mapsto$

$$\mathrm{Hom}_{\mathcal{O}_Y}\left[\omega_Y, f_*\left((\omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y}) \otimes_{\mathcal{D}_X} M\right)\right] \cong$$

$$\omega_Y^\vee \otimes_{\mathcal{O}_Y} f_*\left((\omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y}) \otimes_{\mathcal{D}_X} M\right) \cong$$

$$f_*\left[\underbrace{f^{-1}\omega_Y^\vee \otimes_{\mathcal{O}_Y} (\omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y})} \otimes_{\mathcal{D}_X} M\right]$$

$$\cong \underbrace{\mathrm{Hom}_{f^{-1}\mathcal{O}_Y}\left(f^{-1}\omega_Y, \underbrace{\omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y}}_{\substack{\text{right } \mathcal{D}_X\text{-mod} \\ \text{right } f^{-1}\mathcal{D}_Y\text{-mod}}}\right)}_{\text{left } f^{-1}\mathcal{D}_Y\text{-mod, also right } \mathcal{D}_X\text{-mod}} =: \mathcal{D}_{Y\leftarrow X}$$

$\rightarrow$ $(f^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-bimodule

$$M \mapsto f_*\left(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} M\right) \text{ direct image:}$$

$$\mathrm{Mod}(\mathcal{D}_X) \to \mathrm{Mod}(\mathcal{D}_Y)$$

<u>local descriptions</u>: $i : X \hookrightarrow Y$ embedding

locally given by $(x_1,\dots,x_r) \mapsto (x_1,\dots,x_r,0\dots 0) =: (y_1\dots y_n)$

Then $\mathcal{D}_{Y\leftarrow X} = \mathbb{C}[\partial_{y_{r+1}},\dots,\partial_{y_n}] \otimes_{\mathbb{C}} \mathcal{D}_X = i^{-1}\mathcal{D}_Y / i^{-1}\mathcal{D}_Y(y_{r+1},\dots,y_n)$

<u>ex</u>: (a) prove this, (b) make bimodule structure explicit.

<u>case of projection</u>: $f : X = Y \times T \to Y$, $(x_1,\dots,x_n) \mapsto (x_1,\dots,x_r) =: (y_1\dots y_r)$. Then

$$\mathcal{D}_{Y\leftarrow X} \simeq \mathbb{C}[\partial_{y_1},\dots,\partial_{y_r}] \otimes_{\mathbb{C}} \mathcal{O}_X \simeq \mathcal{D}_X / (\partial_{x_{r+1}},\dots,\partial_{x_n})\cdot\mathcal{D}_X$$

<u>ex</u>: as above

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for a proper definition of direct images $\rightsquigarrow$ need to work in derived categories. Some preliminaries:

We work with $\mathrm{Mod}_{q.c.}(\mathcal{D}_X) = \{ M \in \mathrm{Mod}(\mathcal{D}_X) \mid M \text{ is } \mathcal{O}_X\text{-quasi-coherent}\}$

<u>Def</u>: X smooth alg. variety is called D-affine iff

1.) $\Gamma(X,-): Mod_{q.c}(\mathcal{D}_X) \to Mod(\Gamma(X,\mathcal{D}_X))$ is exact

2.) $\forall \mathcal{M} \in Mod_{q.c}(\mathcal{D}_X): \Gamma(X,\mathcal{M})=0 \Rightarrow \mathcal{M}=0$

Lemma: (exercise) 1.) X affine $\Rightarrow X$ D-affine

2.) X D-affine $\Rightarrow \exists$ equivalence of cat's:

$$\Gamma(X,-): Mod_{q.c.}(\mathcal{D}_X) \to Mod(\Gamma(X,\mathcal{D}_X))$$

HOWEVER (later): $\mathbb{P}^n$ is D-affine

We have subcategory $Mod_c(\mathcal{D}_X) \doteq \{\mathcal{M} \in Mod_{q.c}(\mathcal{D}_X) \mid \mathcal{M}$ locally finitely generated$\}$

Theorem: $\mathcal{M} \in Mod(\mathcal{D}_X)$ is coherent over $\mathcal{O}_X \iff \mathcal{M}$ is $\mathcal{O}_X$-locally free, i.e. an integrable connection

Proof: to show: Let $\mathcal{M} \in Mod(\mathcal{D}_X)$ be $\mathcal{O}_X$-coherent, then $\forall x \in X$: $\mathcal{M}_x$ is a free $\mathcal{O}_{X,x}$-module. Choose local coordinates $(x_1,\dots,x_n)$ at x (in part.: $m_{X,x} = (x_1,\dots,x_n)$)

Take a $\mathbb{C}$-basis $\overline{m_1}, \dots, \overline{m_k}$ of $M_x / \sum_{i=1}^{n} x_i \cdot M_x$

$\xRightarrow{\text{Nakayama}}$ M_x is $\mathcal{O}_{X,x}$-generated by $m_1, \dots, m_k$.

Suppose: $\exists$ non-trivial relation: $\sum_{i=1}^{k} f_i \cdot m_i \overset{(*)}{=} 0$, $f_i \in \mathcal{O}_{X,x}$

Put $\operatorname{ord}(f_i) := \max(\ell \mid f_i \in m^\ell)$. Apply ∂_{x_j} to $(*)$:

$$0 = \sum_{i=1}^{k} \Big[(\partial_{x_j} f_i) \cdot m_i + f_i \cdot \underbrace{(\partial_{x_j} m_i)}_{M_x = \sum_{i=1}^{k} \mathcal{O}_{X,x} m_i} \Big] = \sum_{i=1}^{k} g_i^{(j)} \cdot m_i \quad \text{new relation.}$$

Key point: $\exists j : \min_{i=1..k} \operatorname{ord}(f_i) > \min_{i=1..k} \operatorname{ord}(g_i^{(j)})$

since $\operatorname{ord}(\partial_{x_j} f_i) < \operatorname{ord}(f_i)$ $\forall i,j$ and $f_i \cdot \partial_{x_j} m_i =$ $f_i \cdot \sum \hat{g}_r^{(j)} \cdot m_r$ and $\operatorname{ord}(f_i \cdot \hat{g}_r^{(j)}) \geq \operatorname{ord}(f_r)$.

hence $\exists$ relation $\sum_{i=1}^{k} h_i \cdot m_i = 0$ s.t. $\exists j : \operatorname{ord}(h_j) = 0$

$\Rightarrow \sum \overline{h_i} \cdot \overline{m_i} = 0$ is non-trivial relation in $M_x / m_{X,x} M_x$

$\lightning$

$\square$