Lecture 2: Differential operators, left & right D-modules

X smooth alg. variety $/ \mathbb{C}$ of dim $= n$

$\mathcal{O}_X$ structure sheaf ($f \in \mathcal{O}_X$ for $f \in \mathcal{O}_X(V)$ for <u>some</u> $V \subset X$ open). Let

$\Theta_X = \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X) := \{ v \in \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X) \mid v(fg) = v(f)g + fv(g) \}$

be the tangent sheaf. We have naturally

$$\mathcal{O}_X \hookrightarrow \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X), \ f \mapsto g \mapsto fg$$

<u>Def.</u>: $\mathcal{D}_X$ subsheaf of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$ generated by $\mathcal{O}_X, \Theta_X$: <u>sheaf of differential operators on X</u>.

$\forall x \in X$: $\exists\, \overset{\ni x}{U} \subset X$ affine open s.t.:

- $\exists\, x_1, \dots, x_n \in \mathfrak{m}_{\mathcal{O}_{X,x}}$: $(x_1, \dots, x_n) = \mathfrak{m}_{\mathcal{O}_{X,x}}$, + extend to $\mathcal{O}_X(U)$
- $\Omega^1_X(U) = \bigoplus \mathcal{O}_X(U)\, dx_i$

Then: let $\partial_{x_1}, \dots, \partial_{x_n}$ dual basis of $\Theta_X(U) = Hom_{\mathcal{O}_X(U)}(\Omega^1_X(U), \mathcal{O}_X(U))$

- $[\partial_{x_i}, \partial_{x_j}] = 0$ $\left([\partial_{x_i}, \partial_{x_j}] = \sum c_{ij}^k \partial_{x_k} \Rightarrow c_{ij}^k = [\partial_{x_i}, \partial_{x_j}](x_k) = \partial_{x_i}\partial_{x_j} x_k - \partial_{x_j}\partial_{x_i} x_k = 0 \right)$

- $\mathcal{D}_X(U) = \bigoplus_{\underline{\alpha} \in \mathbb{N}^n} \mathcal{O}_X(U) \cdot \partial_{x_1}^{\alpha_1} \cdot \dots \cdot \partial_{x_n}^{\alpha_n}$ $\qquad \mathbb{N} = \mathbb{N}_0$

call such an $(U, \underline{x})$ a local coordinate-system. Notice: $\underline{x} : U \to \mathbb{C}^n$ gives only étale map.

<u>order filtration</u>: $U \subset X$ affine with coord. system $\underline{x}$. Define:

$$F_k \mathcal{D}_X(U) = \sum_{|\alpha| \leq k} \mathcal{O}_X(U) \cdot \partial_{\underline{x}}^{\alpha}$$ and for general $V \subset X$ open:

$$F_k \mathcal{D}_X(V) := \{ P \in \mathcal{D}_X(V) \mid P_{|U} \in F_k \mathcal{D}_X(U) \ \forall \ (U, \underline{x}),\ U \text{ affine with local coord. system } \underline{x} \}$$

$F_k \mathcal{D}_X = 0 \quad \forall k < 0$

Facts (ex.): 1.) $F_k \mathcal{D}_X \subset F_{k+1} \mathcal{D}_X$,

$\mathcal{D}_X = \bigcup_{k \in \mathbb{N}} F_k \mathcal{D}_X$, $F_k \mathcal{D}_X$ is $\mathcal{O}_X$-locally free

2.) $F_0 \mathcal{D}_X = \mathcal{O}_X$, $F_k \mathcal{D}_X \cdot F_e \mathcal{D}_X = F_{k+e} \mathcal{D}_X$

$(\mathcal{D}_X, F)$ is filtered ring

3.) $P \in F_k \mathcal{D}_X$, $Q \in F_e \mathcal{D}_X$ (local sections)

$\Longrightarrow [P, Q] \in F_{k+e-1} \mathcal{D}_X$

4.) $F_k \mathcal{D}_X = \left\{ P \in \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X) \;\middle|\; [P, f] \in F_{k-1} \mathcal{D}_X \ \forall f \in \mathcal{O}_X \right\}$

this $+ \mathcal{D}_X = \bigcup F_k \mathcal{D}_X$ is Grothendieck's definition of $\mathcal{D}_X$, makes also sense for X singular.

Consider $\mathrm{gr}^F_\bullet \mathcal{D}_X = \bigoplus_{k=0}^{\infty} F_k\mathcal{D}_X / F_{k-1}\mathcal{D}_X$, $F_{-1}\mathcal{D}_X = 0$

For $P \in F_k\mathcal{D}_X(U) \setminus F_{k-1}\mathcal{D}_X(U)$, write $\sigma(P) \in F_k\mathcal{D}_X(U) / F_{k-1}\mathcal{D}_X(U)$: principal symbol.

$\mathrm{gr}^F_\bullet \mathcal{D}_X$ is graded (sheaf of) rings (since $(\mathcal{D}_X, F_\bullet)$ is filtered (sheaf of) rings). Lemma (ex.): $\mathrm{gr}^F_\bullet \mathcal{D}_X$ is commutative (use 3.) above). Locally on $(U, \underline{x})$: Put $\xi_i := \sigma(\partial_{x_i}) \in \mathrm{gr}_1 \mathcal{D}_X(U)$, then

$$\mathrm{gr}^F_k \mathcal{D}_X(U) = F_k\mathcal{D}_X(U) / F_{k-1}\mathcal{D}_X(U) = \bigoplus_{|\alpha|=k} \mathcal{O}_X(U)\,\xi^\alpha$$

and so $\mathrm{gr}^F_\bullet \mathcal{D}_X(U) = \mathcal{O}_X(U)[\xi_1, \dots, \xi_n]$

<u>This globalizes:</u> Consider cotangent bundle $\pi : T^*X \to X$, then locally

$$\mathcal{H}om_{\mathcal{O}_X}(T^*X, \mathcal{O}_X)(U) \cong \bigoplus_{i=1}^{n} \mathcal{O}_X(U) \cdot \xi_i$$

$$\Longrightarrow \underline{Sym}_{\mathcal{O}_X}(\Theta_X) \cong Sym_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(T^*X, \mathcal{O}_X))$$

$$\cong gr^F_\bullet \mathcal{D}_X \quad \text{globally}$$

But $\underline{Sym}_{\mathcal{O}_X}(\Theta_X) \cong \pi_* \mathcal{O}_{T^*X}$

hence $gr^F_\bullet \mathcal{D}_X \cong \pi_* \mathcal{O}_{T^*X} \in QCoh(\mathcal{O}_X)$.

<u>Def:</u> A differential system on X is a sheaf $\mathcal{M}$ of left $\mathcal{D}_X$-modules.

Can rephrase this in the language of connections

<u>Lemma:</u> Let $\mathcal{M}$ be an $\mathcal{O}_X$-module.

Then a $\mathcal{D}_X$-module structure on $\mathcal{M}$

$\Leftrightarrow \exists \nabla : \Theta_X \to \mathcal{E}nd_{\mathbb{C}}(\mathcal{M})$, $v \mapsto \nabla_v$ s.t.

$$\left.\begin{array}{l} \nabla_{fv}(s) = f\nabla_v(s) \\ \nabla_v(fs) = f\nabla_v(s) + v(f)\nabla_v(s) \\ \nabla_{[v,g]}(s) = [\nabla_v, \nabla_g](s) \end{array}\right\} \begin{array}{l} \forall f \in \mathcal{O}_X, \\ v, g \in \Theta_X \\ s \in \mathcal{M} \end{array}$$

Pf follows from definition of $\mathcal{D}_X$ & relation $[v, f] = v(f)$ in $\mathcal{D}_X$

$\Leftrightarrow \exists \nabla : \mathcal{M} \to \Omega^1_X \otimes \mathcal{M}$ $\mathbb{C}$-lin s.t. $\nabla(fs) = f\nabla s + df \otimes s$

& s.t. $\nabla^{(2)} \circ \nabla = 0$, where $\nabla^{(2)} : \Omega^1_X \otimes \mathcal{M} \to \Omega^2_X \otimes \mathcal{M}$

$$\alpha \otimes s \mapsto d\alpha \otimes s - \alpha \wedge \nabla s.$$

<u>Pf.</u>: exercise

Def.: Call a $\mathcal{D}_X$-module $\mathcal{M}$ an integrable connection iff $\mathcal{M}$ is $\mathcal{O}_X$-locally free.

<u>example:</u> Let $P = a_n(t)\partial_t^n + \ldots + a_0(t)$, $a_i \in \mathbb{C}[t]$ an ordinary diff. equation, put $M = D_{\mathbb{C}} / D \cdot P$ write $u = [1]$ (class of 1 in M), then $M = D \cdot u$.

Put $U = \{ t \in \mathbb{A}^1_{\mathbb{C}} \mid a_n(t) \neq 0 \}$, then

$$M_{|U} \simeq \bigoplus \mathcal{O}_U \cdot u^{(i)} \quad \text{with } u^{(i)} = \partial_t^i \cdot u$$

<u>Pf:</u> One checks that in $\Gamma(U, \mathcal{D}_{\mathbb{C}}) = \mathbb{C}[t][\frac{1}{a_n}]\langle \partial_t \rangle$ $\exists$ division theorem by P: $\forall Q \in \Gamma(U, \mathcal{D}_{\mathbb{C}})$: $\exists S$: $Q = S \cdot P + R$, $\deg R < n$. Hence M is generated by $u^{(i)}$, and actually free $/ \mathcal{O}_U$.

Left- and right $\mathcal{D}_X$-modules:

For operator $P = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x) \partial_x^\alpha \in \mathcal{D}_X(U)$, define its transpose or adjoint:

$$^tP := \sum (-\partial_x)^\alpha \cdot a_\alpha(x)$$

Then: $\forall P, Q \in \mathcal{D}_X(U)$: $^t(P \cdot Q) = {}^tQ \cdot {}^tP$.

If $M(U)$ is left $\mathcal{D}_X(U)$-module, put $s \cdot P := {}^tP \cdot s$ $\forall$ $P \in \mathcal{D}_X(U)$, $s \in M(U)$, this defines right $\mathcal{D}_X(U)$-mod. structure on $M(U)$. We want to globalize this!

Recall that for $v \in \Theta_X$, $\omega \in \Omega_X^k = \Lambda^k \Omega_X$, we have the Lie derivative $\mathrm{Lie}_v \omega \in \Omega_X^k$ defined by:

$$(\mathrm{Lie}_v \omega)(\theta_1, \dots, \theta_k) := v(\omega(\theta_1, \dots, \theta_k)) - \sum_{i=1}^n \omega(\theta_1, \dots, [v, \theta_i], \dots, \theta_k)$$

We have:

1.) $Lie_v(f\omega) = v(f)\omega + f\, Lie_v(\omega)$

2.) $Lie_{[v_1, v_2]}(\omega) = [Lie_{v_1}, Lie_{v_2}](\omega)$

3.) Cartan's formula: $Lie_v = d i_v + i_v d$.

<u>Notice</u>: (4) $k = n \Rightarrow Lie_{f\cdot v}(\omega) = d i_{fv}\omega + i_{fv}\underbrace{d\omega}_{=0}$

$$= d i_v(f\omega) = d i_v(f\omega) + i_v \underbrace{d(f\omega)}_{0} = Lie_v(f\omega)$$

Put $\omega_X := \Lambda^n \Omega^1_X$ (canonical bundle)

then $\omega_X \times \Theta_X \to \omega_X : \omega.\theta := -Lie_\theta \omega$ yields

right $\mathcal{D}_X$-mod. structure on ω_X.

(Pf uses (4): exercise)

<u>ex.</u>: If (U, x) is local coordinate system:

$\omega = f(x)\cdot dx_1 \wedge \dots \wedge dx_n \in \omega_X$, $P \in \mathcal{D}_X(U)$:

$\omega \cdot P = ({}^t P f)\, dx_1 \wedge \dots \wedge dx_n$

more generally: $\mathcal{M}$ $\mathcal{O}_X$-module then right $\mathcal{D}_X$-module structure on $\mathcal{M}$ $\iff$

$\nabla' : \Theta_X \to \mathrm{End}_{\mathbb{C}}(\mathcal{M})$ s.t.

$$\left.\begin{aligned} \nabla'_{fv}(s) &= \nabla'_v(fs) \\ \nabla'_v(fs) &= f\nabla'_v(s) + v(f)\nabla'_v(s) \\ \nabla'_{[v_1,v_2]} &= [\nabla'_{v_1}, \nabla'_{v_2}] \end{aligned}\right\} \quad \forall f \in \mathcal{O}_X,\ v, v_1, v_2 \in \Theta_X,\ s \in \mathcal{M}$$

ex. similarly to the left case

Aim: correspondence left $\mathcal{D}_X$ <-> right $\mathcal{D}_X$-modules

Prop.: (a) $\mathcal{M}, \mathcal{N}$ left $\mathcal{D}_X$-modules $\Rightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ is left module by $v \mapsto (m \otimes n \mapsto vm \otimes n + m \otimes vn)$

(b) $\mathcal{M}'$ right $\mathcal{D}_X$-mod, $\mathcal{N}$ left $\mathcal{D}_X$-mod $\Rightarrow \mathcal{M}' \otimes_{\mathcal{O}_X} \mathcal{N}$ is right mod by $v \mapsto (m \otimes n \mapsto mv \otimes n + m \otimes vn)$

$fv \mapsto [m \otimes n \mapsto (m(fv)) \otimes n + m \otimes (fv)n) = ((mf)v \otimes n + (mf) \otimes vn)$
$= (mf \otimes n).v = (m \otimes n)f.v]$

(notice: tensor product of 2 right str. is NOT right module)

(c) $\mathcal{M}, \mathcal{N}$ left $\mathcal{D}_X$-mod $\Rightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ is left $\mathcal{D}_X$-module by:

$$v \mapsto \Big(\varphi \mapsto \big(m \mapsto v(\varphi(m)) - \varphi(v(m))\big)\Big)$$

(d) $\mathcal{M}', \mathcal{N}'$ right $\mathcal{D}_X$-mod $\Rightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ is left (!) -mod by:

$$v \mapsto \Big(\varphi \mapsto \big(m \mapsto \varphi(m \cdot v) - \varphi(m) \cdot v\big)\Big)$$

(e) $\mathcal{M}$ left $\mathcal{D}_X$-mod, $\mathcal{N}'$ right $\mathcal{D}_X$-mod $\Rightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}')$ right $\mathcal{D}_X$-mod by

$$v \mapsto \Big(\varphi \mapsto \big(m \mapsto \varphi(m) \cdot v + \varphi(vm)\big)\Big)$$

$$fv \mapsto \Big(\varphi \mapsto \big(m \mapsto \varphi(m)(fv) + \varphi((fv) \cdot m)$$

$$= (\varphi(m) \cdot f) v + \underbrace{\varphi(f\, vm)}_{\varphi(vm) \cdot f}$$

$$= (f\varphi)(m) v + (f\varphi)(vm)$$

$$= (f\varphi \cdot v)(m)\big)\Big)$$

<u>Corollary:</u> $\mathcal{M}$ left $\mathcal{D}_X$-module, then

$\mathcal{M}' := \omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is right $\mathcal{D}_X$-module

(associated right module)

$\mathcal{N}'$ right module $\Rightarrow \mathcal{N} := \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{N}')$

is left $\mathcal{D}_X$-module.

This is inverse to each other, i.e.

$$\mathcal{M} \simeq \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{M})$$

$$\mathcal{N}' \simeq \omega_X \otimes_{\mathcal{O}_X} \mathcal{H}om(\omega_X, \mathcal{N}')$$

as left / right $\mathcal{D}_X$-mod

(Notice $\forall \mathcal{G} \in VB : \mathcal{G}^\vee \otimes_{\mathcal{O}} \mathcal{F} \simeq \mathcal{H}om(\mathcal{G}, \mathcal{F})$

$\Rightarrow \mathcal{H}om(\omega_X, \mathcal{N}') \simeq \Theta_X \otimes \mathcal{N}' \dots$)

Aim now: For a morphism $f: X \to Y$

define direct & inverse image

Inverse images: M left $\mathcal{D}_Y$-module

Consider $f^*M := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}M$ as $\mathcal{O}_X$-module. Put left $\mathcal{D}_X$-structure on this:

we have tangent map $Tf : \Theta_X \to f^*\Theta_Y$, def. by: $v \mapsto (g_{\in \mathcal{O}_Y} \mapsto v(g \circ f))$ (dual to morphism $f^*\Omega^1_Y \to \Omega^1_X$ appearing in cotangent sequence $f^*\Omega^1_X \to \Omega^1_X \to \Omega^1_{X/Y} \to 0$)

Put $\Theta_X \to \mathcal{E}nd_{\mathbb{C}}(f^*M)$, $v \mapsto \Big(\underset{\in \mathcal{O}_X}{h} \otimes \underset{\in f^{-1}M}{m} \mapsto v(h) \otimes m + h \otimes Tf(v)(m)\Big)$

<u>ex:</u> this defines left $\mathcal{D}_X$-mod-str. on f^*M

<u>alternative description</u>: Take $\mathcal{D}_Y$ as left module over itself and apply above construction to it

$\Rightarrow \mathcal{D}_{X\to Y} := f^*\mathcal{D}_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$

as we have seen, this has left $\mathcal{D}_X$-mod. structure. On the other hand, we have "trivial" right $f^{-1}\mathcal{D}_Y$-structure: $(h \otimes P)Q := h \otimes PQ$

$(\forall h \in \mathcal{O}_X,\ P, Q \in f^{-1}\mathcal{D}_Y)$

<u>Def.</u>: The $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bimodule $\mathcal{D}_{X\to Y}$ is called transfer module

<u>Fact</u>: $\forall \mathcal{M} \in \mathrm{Mod}(\mathcal{D}_Y)$, we have

$$f^*\mathcal{M} = \mathcal{O}_X \underset{f^{-1}\mathcal{O}_Y}{\otimes} f^{-1}\mathcal{M} \overset{!}{=} \underbrace{\mathcal{D}_{X\to Y} \underset{f^{-1}\mathcal{D}_Y}{\otimes} f^{-1}\mathcal{M}}$$

tensor product of a right $f^{-1}\mathcal{D}_Y$-module with a left $f^{-1}\mathcal{D}_Y$-mod. inherits left $\mathcal{D}_X$-str.

<u>Hence</u>: $\mathcal{D}_{X\to Y} \underset{f^{-1}\mathcal{D}_Y}{\otimes} f^{-1}(-) : \mathrm{Mod}(\mathcal{D}_Y) \longrightarrow \mathrm{Mod}(\mathcal{D}_X)$ (left mod.)

right exact functor

local expression for closed embeddings:

let $i : X \hookrightarrow Y$ closed emb. with X, Y smooth.

for $x \in X$, suppose that we have local coordinates $y_1, \dots, y_n$ of Y on $x \in U \subset Y$ s.t. $X = \{y_{r+1} = \dots = y_n = 0\}$. Put $x_i := y_i \circ i$ $\forall i = 1, \dots r$

$\rightsquigarrow$ local coord. on $V \subset X$ near x. Tangent map

$$\Theta_V \to i^* \Theta_U = \mathcal{O}_V \otimes_{f^{-1}\mathcal{O}_U} f^{-1}\Theta_U \text{ is } \partial_{x_i} \mapsto \partial_{y_i} \quad i = 1, \dots, r$$

write $\mathcal{D}' := \bigoplus_{\alpha_1, \dots, \alpha_r} \mathcal{O}_U \partial_{y_1}^{\alpha_1} \cdots \partial_{y_r}^{\alpha_r} \subset \mathcal{D}_U$ subring

as left $\mathcal{D}'$-mod.

and we have $\mathcal{D}_U \overset{\downarrow}{\simeq} \mathcal{D}' \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}]$ $\left(\simeq \bigoplus_{\alpha_1 \dots \alpha_n} \mathcal{O}_U \partial_{y_1}^{\alpha_1} \cdots \partial_{y_n}^{\alpha_n} \right)$

$$\Rightarrow \mathcal{D}_{V \to U} = \mathcal{O}_V \otimes_{f^{-1}\mathcal{O}_U} f^{-1}\mathcal{D}_U = \mathcal{O}_V \otimes_{f^{-1}\mathcal{O}_U} \left(\mathcal{D}' \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}] \right)$$

$$\simeq \underbrace{\left(\mathcal{O}_V \otimes_{f^{-1}\mathcal{O}_U} \mathcal{D}' \right)}_{\substack{\wr\wr \\ \mathcal{D}_V = \bigoplus \mathcal{O}_V \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}}} \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}] \simeq \mathcal{D}_V \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}]$$

iso of left $\mathcal{D}_V$-modules

$\Rightarrow \mathcal{D}_{X \to Y}$ left $\mathcal{D}_X$-locally free $\left(\substack{\text{rk} = \infty \\ \text{in gen.}} \right)$