Irregular Hodge numbers of Frenkel–Gross connections

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December 8, 2024

Abstract

Frenkel and Gross constructed a family of connections on $\mathbb{P}^1\backslash\{0,\infty\}$, for almost simple groups $\check{G}$ and their representations. In this article, we calculate the irregular Hodge numbers of these Frenkel–Gross connections, and, as an application, we prove a conjecture of Katzarkov–Kontsevich–Pantev for mirror Landau-Ginzburg models of minuscule homogeneous spaces.

Contents

1 Introduction

The study of irregular singularities of systems of linear differential equations has gained increasing importance across various mathematical fields, such as Hodge theory or arithmetics. Since the differential equations underlying variations of Hodge structures (VHS) can only have regular singularities, we need a broader framework to discuss the Hodge theoretic properties of irregular connections.

A famous example is the Bessel differential equation $(t\partial_t)^{n+1} - t$. The associated connection, called the *Bessel connection* (or *Kloosterman connection*), is the connection

$$\nabla = \mathrm{d} + \begin{pmatrix} 0 & & & t \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \frac{\mathrm{d}t}{t} \tag{1.0.1}$$

on the trivial bundle $\mathcal{O}_{\mathbb{G}_m}^{n+1}$, denoted by Be_{n+1}. It has a regular singularity at 0, and an irregular singularity at ∞ of slope $\frac{1}{n+1}$. When $n = 1$, Deligne showed that Be_2 cannot underlie a variation of Hodge structures [Del07, §8], and hence, there does not exist any Hodge filtration on Be_2. Instead, Deligne suggested and studied the concept of irregular Hodge filtrations on irregular connections [Del07], which was further developed by Esnault, Kontsevich, Sabbah, and Yu in [Sab10, Yu14, SY15, ESY17]. For example, the

2020 *Mathematics subject classification.* 32C38, 32S40, 20G20, 14M17, 14D07

Key words. Frenkel-Gross connections, Rescalable Twistor $\mathscr{D}$-modules, Irregular Hodge filtration, Landau–Ginzburg models.

irregular Hodge filtration on Be_{n+1} jumps at $0, 1, \ldots, n$, and the irregular Hodge numbers are all 1, see for example [CnDS21].

Since then, a solid framework of the theory of irregular Hodge structures and irregular Hodge modules has been developed in [KS11, Sab18a, FSY22, Moc21]. We will use the fundamental results of these papers in an essential way here. Rather than using the term irregular Hodge modules, we adopt the terminology of rescalable integrable mixed twistor $\mathscr{D}$-module, as introduced in [Moc21], cf. Section 3.2. In particular, the underlying $\mathscr{D}$-module of any rescalable integrable mixed twistor $\mathscr{D}$-module comes equipped with an irregular Hodge filtration.

The purpose of this paper is to study irregular Hodge theory for certain irregular connections on principal bundles. More precisely, given any almost simple complex algebraic group $\check{G}$, we are concerned with a family of connections (denoted $\nabla_{\check{G}}$) on the trivial $\check{G}$-bundle over $\mathbb{P}^1 \backslash \{0, \infty\}$ that has been constructed by Frenkel and Gross in [FG09]. These connections are regular singular at 0 and irregular singular at ∞. By considering specific representations of $\check{G}$, one can recover examples such as the Bessel connection and certain hypergeometric connections, cf. Example 2.1.

Our main result establishes the existence of a rescalable integrable mixed twistor $\mathscr{D}$-module associated with Frenkel–Gross connections and provides an explicit computation of the corresponding irregular Hodge numbers.

In the statement of the following theorem, we denote by $\rho = 2\rho/2$ the half sum of positive coroots of $\check{G}$, which can also be considered as the half sum of positive roots of G. We recall a few more facts and notations around algebraic groups at the end of this introduction.

Theorem 1.1. *For an almost simple reductive group $\check{G}$ over $\mathbb{C}$, the Frenkel–Gross connection $\nabla_{\check{G}}$ underlies a unique rescalable integrable mixed twistor $\mathscr{D}$-module $\nabla^{\mathrm{H}}_{\check{G}}$. The corresponding irregular Hodge numbers are determined*[1] *by $\rho = 2\rho/2$, the half sum of positive coroots of $\check{G}$. Concretely, for each representation V of $\check{G}$, the irregular Hodge numbers $h^{\alpha}_{\mathrm{irr}}$ of $\nabla_{\check{G}}(V)$ are the dimensions of the eigenspaces for the eigenvalues 2α of V under the $\mathbb{G}_m$-action $\mathbb{G}_m \xrightarrow{2\rho} \check{G} \to \mathrm{GL}(V)$.*

Remark 1.2. One of Deligne's motivations for studying irregular Hodge filtrations is to obtain lower bounds for p-adic slopes of exponential sums. Let us mention an arithmetic application of Theorem 1.1 for the case of Frenkel–Gross connections, as mentioned in [LT24, Rem. 13.9]. In [XZ22, Thm. 5.3.2], Xu and Zhu equipped Frenkel–Gross connections with some Frobenius structures and studied their p-adic slopes —i.e., p-adic valuations of eigenvalues of Frobenius on Frenkel–Gross connections. The traces of Frobenius are called *Kloosterman sums (for reductive groups)*, which are sums of these eigenvalues. Xu and Zhu showed that the p-adic slopes are also determined by ρ (except for finitely many points). Combining their result with Theorem 1.1 yields that the p-adic slopes agree with the irregular Hodge numbers, which generalizes a theorem of Sperber [Spe80]. ◇

Another motivation for studying irregular Hodge filtration arises from mirror symmetry. As mirrors of Fano varieties one considers Landau–Ginzburg models, namely, pairs (Y, w) consisting of a quasi-projective complex variety and a regular function $w\colon Y \to \mathbb{A}^1$. A great deal of work has been carried out over the last decades to establish such miror correspondences, see, e.g., [Iri09] and [RS15, RS17] for the toric case, and, on the other hand, [Rie08] and [LT24] for the case of homogeneous spaces.

In [KKP17], Katzarkov–Kontsevich–Pantev proposed three kinds of (conjecturally identical) Hodge numbers of (Y, w), including the irregular Hodge numbers (notice that one of these conjectured equalities is incorrectly stated though, as can be seen by looking at the case of the mirror of projective spaces). In particular, this yields a conjectured a relationship comparing the Hodge numbers for Fano varieties and the irregular Hodge numbers for their Landau–Ginzburg models. As an application of Theorem 1.1, we confirm this conjecture in Theorem 4.1 for the Fano varieties that are minuscule homogeneous spaces.

The irregular Hodge filtration (as opposed to the irregular Hodge numbers, i.e., the dimension of the graded pieces of this filtration) is not a priori controlled by our main result. However, we point out in Section 6 a conjecture that, if shown, would lead to an alternative proof of our main result and would give, as an additional benefit, that the irregular Hodge filtration itself is also determined by the cocharacter ρ (in the sense of Definition 3.5).

[1] See Definition 3.5 for a precise definition.

Idea of the proof of Theorem 1.1 In order to orient the reader, let us sketch informally the strategy to show our main result Theorem 1.1. The first ingredient is the fact that the Frenkel–Gross connections $\nabla_{\check{G}}(V)$ are of (exponentially) geometrical origin. In fact, by a theorem of Zhu [Zhu17a], they are eigen $\mathscr{D}$-modules of Hecke operators, constructed via the method of Heinloth–Ngô–Yun [HNY13]. Using this strategy, we equip in Section 3.2 the connection $\nabla_{\check{G}}(V)$ with the structure of a rescalable integrable pure twistor $\mathscr{D}$-module of weight 0. Furthermore, we show that there is a unique lifting of $\nabla_{\check{G}}$ to a tensor functor $\nabla^{\mathrm{H}}_{\check{G}}$ from the category of finite dimensional representations of $\check{G}$ to the category of rescalable integrable mixed twistor $\mathscr{D}$-modules on $\mathbb{G}_m$ (see the discussion before formula (3.2.5)).

Via the functoriality of $\nabla^{\mathrm{H}}_{\check{G}}$ (see Section 2.2), we reduce (Lemmas 3.13 and 3.14) the calculation of Hodge numbers to that of $\nabla^{\mathrm{H}}_{\check{G}}(V)$ for simple groups $\check{G}$ and $V = \mathrm{Ad}(\check{G})$, the adjoint representation of $\check{G}$.

The next step is to express the Frenkel–Gross connections $\nabla_{\check{G}}(V)$ as the Fourier transforms of some regular singular connections. By the stationary phase principle (cf. [Sab08]), the Fourier transform of a regular $\mathscr{D}_{\mathbb{A}^1_\tau}$-module is a $\mathscr{D}_{\mathbb{A}^1_t}$-module (where $\mathbb{A}^1_t$ is the dual affine line of $\mathbb{A}^1_\tau$) which is smooth on $\mathbb{G}_{m,t}$, regular singular at 0, and irregular singular at ∞ of slope 0 or 1. Frenkel and Gross showed that $\nabla_{\check{G}}(V)$ has slope 0 or $1/h$ where h is the Coxeter number of $\check{G}$. Therefore, we rather work with $\widetilde{\nabla}_{\check{G}}(V) := [h]^+\nabla_{\check{G}}(V)$, which has the same irregular Hodge numbers as $\nabla_{\check{G}}(V)$. Using the geometrical presentation of Frenkel–Gross connections mentioned above, we express $\widetilde{\nabla}_{\check{G}}(V)$ as the Fourier transform of some regular $\mathscr{D}_{\mathbb{A}^1_\tau}$-modules in Proposition 2.5.

By a (filtered) stationary phase formula [SY19], to calculate the irregular Hodge numbers of $\widetilde{\nabla}_{\check{G}}(V)$, it suffices to calculate the Hodge numbers of the nearby cycle at infinity of $\mathrm{FT}^{-1}(j_+\nabla^{\mathrm{H}}_{\check{G}}(V))$, where $j : \mathbb{G}_{m,t} \hookrightarrow \mathbb{A}^1_t$. In general, the inverse Fourier transform of Frenkel–Gross connections involves some intersection complexes, which makes the Hodge numbers tricky to handle. However, the nilpotent part of the monodromy of the Frenkel–Gross connections at 0 is quite special, namely principally nilpotent. Through the stationary phase principle mentioned above, we can easily determine the monodromy of the inverse Fourier transform $\mathrm{FT}^{-1}\widetilde{\nabla}_{\check{G}}(V)$ at ∞ using representation-theoretic information of V. In particular, when the nilpotent part of the monodromy operator can be decomposed into Jordan blocks of different sizes, we calculate in Corollary 3.16 the formula of Hodge numbers of the nearby cycle of $\mathrm{FT}^{-1}(j_+\nabla^{\mathrm{H}}_{\check{G}}(V))$ at ∞. Since this somewhat artificial condition is always true for adjoint representations, we are able to conclude the proof of our main result.

Plan of the paper Let us give a brief overview on the content of the paper: In Section 2, we collect basic properties of Frenkel–Gross connections. In Section 3, we recall preliminaries on rescalable integrable mixed twistor $\mathscr{D}$-modules and equip Frenkel–Gross connections with such structures. Moreover, we study the inverse Fourier transforms of $\widetilde{\nabla}_{\check{G}}$ in Section 3.3 and prove Theorem 1.1 in Section 3.4. In Section 4, we verify a conjecture of Katzarkov–Kontsevich–Pantev for minuscule flag varieties. In Section 5, we provide concrete calculations of irregular Hodge numbers for certain examples. In Section 6, we make a conjecture of the shape of the underlying $\mathscr{R}$-modules of $\nabla^{\mathrm{H}}_{\check{G}}$.

We finish this introduction by fixing some notation that is used throughout this article:

$\mathscr{D}$-modules Throughout the paper, we work with algebraic $\mathscr{D}$-modules. One may consult [HTT08] as a reference for the notation and results we use. For a smooth variety X over $\mathbb{C}$, we write $D^b_{qc}(\mathscr{D}_X)$ resp. $D^b_h(\mathscr{D}_X)$ for the derived category of quasi-coherent resp. of holonomic $\mathscr{D}_X$-modules. For a morphism $f : X \to Y$ we have functors $f_+ \colon D^b_{qc}(\mathscr{D}_X) \to D^b_{qc}(\mathscr{D}_Y)$ and $f^+ \colon D^b_{qc}(\mathscr{D}_Y) \to D^b_{qc}(\mathscr{D}_X)$. Moreover, there is the duality functor $\mathbb{D} \colon D^b_h(\mathscr{D}_X) \to D^b_h(\mathscr{D}_X)$ that respects $\mathrm{Mod}_h(\mathscr{D}_X)$. We also define functors $f_\dagger := \mathbb{D} \circ f_+ \circ \mathbb{D}$ and $f^\dagger := \mathbb{D} \circ f^+ \circ \mathbb{D}$.

We will use at various place the Fourier transformation functor for algebraic $\mathscr{D}$-modules. The most general definition is when we are starting with a vector bundle $E \to X$, where X is still smooth. Then we have functors $\mathrm{FT}^{\pm 1}_X : D^b(\mathscr{D}_E) \to D^b(\mathscr{D}_{E^\vee})$, where $E^\vee$ is the dual bundle. They are defined by $\mathrm{FT}^{\pm 1}(N) := p_{2,+}(p_1^+ N \otimes \mathcal{E}^{\pm\varphi})$, where $p_1 : E \times E^\vee \to E$ and $p_2 : E \times E^\vee \to E^\vee$ are the projections and where $\varphi \in \mathcal{O}_{E\times E^\vee}$ is the canonical pairing. The functors $\mathrm{FT}^{\pm 1}_X$ are exact (i.e. they send $\mathrm{Mod}(\mathscr{D}_E)$ to $\mathrm{Mod}(\mathscr{D}_{E^\vee})$). Special cases of this definition that we will use are when $X = \{pt\}$ (in which case we write $\mathrm{FT}^{\pm 1} := \mathrm{FT}^{\pm 1}_{pt}$) or when $E \cong \mathbb{A}^1 \times X$ is the trivial line bundle over X (this case is called partial Fourier transformation with respect to the $\mathbb{A}^1$-coordinate).

We consider the abelian category $\mathrm{MHM}(X)$ of algebraic $\mathbb{Q}$-mixed Hodge modules on X (as defined in [Sai88, Sai90]) and we write $D^b\mathrm{MHM}(X)$ for its bounded derived category. Let $f\colon X \to Y$ by any morphism of smooth algebraic varieties, then the functors $f_+, f_\dagger$ resp. $f^\dagger[\dim(Y)-\dim(X)], f^+[\dim(X)-\dim(Y)]$ on $D^b_h(\mathscr{D}_X)$ resp. $D^b_h(\mathscr{D}_Y)$ lift to functors

$$f_*, f_!\colon D^b\mathrm{MHM}(X) \to D^b\mathrm{MHM}(Y) \qquad \text{resp.} \qquad f^*, f^!\colon D^b\mathrm{MHM}(Y) \to D^b\mathrm{MHM}(X).$$

We also denote by $\mathbb{D}$ the functor on $D^b\mathrm{MHM}(X)$ which lifts the above defined holonomic duality functor on $D^b_h(\mathscr{D}_X)$. We also consider $\mathbb{R}$-mixed Hodge modules (as in [Moc15, §13.5]), and we denote the corresponding abelian category by $\mathrm{MHM}(X,\mathbb{R})$ and by $D^b\mathrm{MHM}(X,\mathbb{R})$ the corresponding bounded derived category.

A major part of our considerations will use the category of integrable mixed twistor modules as constructed by Mochizuki (see [Moc15]). We will postpone reminders on the properties of this category to Section 3.1, where the motivation for using this category becomes clear.

Simple groups For an almost simple reductive group G over $\mathbb{C}$, we fix a choice of a maximal torus T and a Borel subgroup B containing T. The *root datum* of G is the data $(X_*, \Phi, X^*, \Phi^\vee, \cdot^\vee)$, where $X^* = X^*(T)$ and $X_* = X_*(T)$ denote respectively the *character* and *cocharacter groups*, where Φ and $\Phi^\vee$ denote the *root system* and *coroot system*, and where $\cdot^\vee : \Phi \to \Phi^\vee : \alpha \mapsto \alpha^\vee$ is a bijection such that $\langle \alpha, \alpha^\vee \rangle = 2$. For each $\alpha \in \Phi$, we denote by $u_\alpha : \mathbb{G}_a \simeq U_\alpha \subset G$ the corresponding *root subgroup*.

With the choice of $T \subset B \subset G$, we have the subset Φ^+ (resp. Φ^-) of positive (resp. negative) roots in Φ, i.e. those $\alpha \in \Phi$ such that $U_\alpha \subset B$ (resp. $U_\alpha \not\subset B$). Moreover, we have a subset $\Delta \subset \Phi^+$ of *simple roots*. Every element in $\Phi^+ = -\Phi^-$ can be expressed as a non-negative integer linear combination of elements in Δ. Moreover, we have the *root space decomposition* (also called the *Cartan decomposition*) of the Lie algebra $\mathfrak{g}$ of G as $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha$, where $\mathfrak{t} = \mathrm{Lie}(T)$ is the Cartan sub-Lie algebra and $\mathfrak{g}_\alpha$ is the (one-dimensional) α-root space for each $\alpha \in \Phi$.

There exists a unique almost simple reductive group $\check{G}$ over $\mathbb{C}$ with root datum $(X^*, \Phi^\vee, X_*, \Phi, (\cdot^\vee)^{-1})$, up to isomorphism, called the *dual group* of G. The corresponding maximal torus and Borel subgroup of G are denoted by $\check{T} \subset \check{B} \subset \check{G}$. In particular, $X^*(\check{T}) = X_*(T)$ and $X_*(\check{T}) = X^*(T)$.

Acknowledgment

We are grateful to Claude Sabbah for answering our questions about various points in the theory of twistor modules. We also thank Hao Sun and Daxin Xu for their valuable discussions.

Y. Qin is supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement no. 101020009, project TameHodge). P. Spacek and Ch. Sevenheck acknowledge support by the DFG-project SE 1114/6-1.

2 Frenkel–Gross connections

In this section, we recall the definition of Frenkel–Gross connections, and discuss their basic properties such as functoriality and their (exponentially) geometrical interpretation.

2.1 Definition of Frenkel–Gross connections

Let $\check{G}$ be an almost simple group over $\mathbb{C}$. As before, we fix a Borel subgroup $\check{B} \subset \check{G}$ and a maximal torus $\check{T} \subset \check{G}$. For each simple root $\alpha_i^\vee$ of $\check{G}$, we denote by $X_{-\alpha_i^\vee}$ a basis vector in the root subspace of $\check{\mathfrak{g}} = \mathrm{Lie}(\check{G})$. Let $E = X_{\theta^\vee}$ be a basis vector in the root subspace of $\check{\mathfrak{g}}$ corresponding to the maximal root $\theta^\vee$. Let $N = \sum_{\alpha^\vee \in \Delta^\vee} X_{-\alpha^\vee}$ be the corresponding principal nilpotent element. Then, the *Frenkel–Gross connection*, denoted by $\nabla_{\check{G}}$, is a trivial G-bundle whose connection is defined by

$$\nabla_{\check{G}} = \mathrm{d} + (N + Et)\,\frac{\mathrm{d}t}{t} \tag{2.1.1}$$

(Notice that the variable t is called z in [FG09], but we are following a different convention here to be consistent with the notation used in the theory of mixed twistor modules, especially with the convention we employ in Section 6). Alternatively, we can view the Frenkel–Gross connection as a tensor functor

$$\nabla_{\check{G}} \colon \mathrm{Rep}(\check{G}) \to \mathrm{Conn}(\mathbb{G}_{m,t}) \tag{2.1.2}$$

from the category of finite-dimensional representations of $\check{G}$ to the category of flat connections on $\mathbb{G}_m$.

It was shown in [FG09] that $\nabla_{\check{G}}$ has a regular singularity at 0 with principal unipotent monodromy in the sense of [Kos59], with N being the nilpotent part of the monodromy at 0. Moreover, it has an irregular singularity at ∞ of slope $\frac{1}{h}$, where $h = h(\check{G})$ is the Coxeter number of $\check{G}$. For the reason of slopes at ∞, we also work with

$$\widetilde{\nabla}_{\check{G}}(V) := [h]^+ \nabla_{\check{G}}(V) \tag{2.1.3}$$

(where $[h] : \mathbb{G}_{m,t} \to \mathbb{G}_{m,t}$ denotes the h-th power map), which has only 0 or 1 as the slopes at ∞, and can also be viewed as a tensor functor.

Moreover, the Frenkel–Gross connections are both cohomologically rigid [FG09, Thm. 1] and physically rigid [Yi24, Thm. 1].

Example 2.1.
- If $\check{G} = \mathrm{SL}_{n+1}$ and $V = \mathbb{C}^{n+1}$ the standard representation of SL_{n+1}, then we recover the Bessel connection in (1.0.1).
- If $\check{G} = \mathrm{SO}_{2n+1}$ and $V = \mathbb{C}^{2n+1}$ the standard representation of SO_{2n+1}, then $\nabla_{\mathrm{SO}_{2n+1}}(V)$ is isomorphic to the hypergeometric connection associated with the differential equation

  $$(t\partial_t)^{2n+1} - 2t(t\partial_t + \tfrac{1}{2}) = 0,$$

  see [FG09, §6].

◇

2.2 Functoriality of Frenkel–Gross connections

The differential Galois groups of $\nabla_{\check{G}}$ are calculated in [FG09, Cor. 9 & 10]. If $\check{G}$ is an almost simple group of a type listed on the left-hand side of the table below, then the differential Galois group G_{gal} is an almost simple group of the type listed on the right-hand side of the same row.

$\check{G}$	G_{gal}
A_{2n}	A_{2n}
A_{2n-1}, C_n	C_n
B_n, $D_{n+1} (n \geq 4)$	B_n
E_7	E_7
E_8	E_8
E_6, F_4	F_4
B_3, D_4, G_2	G_2

(2.2.1)

Let $\check{G}' \subset \check{G}$ be two almost simple groups that appear in the same line in the left column of (2.2.1). Then they have the same differential Galois groups. So we can ask if $\nabla_{\check{G}'}$ is induced by $\nabla_{\check{G}}$. By [XZ22, Thm. 4.3.3 & Prop. 5.13], we have the following result:

Theorem 2.2. *For a choice of N' and E' in $\check{\mathfrak{g}}'$, there exist N and E in $\check{\mathfrak{g}}$ which coincide with N' and E' respectively under the inclusion $\check{\mathfrak{g}}' \subset \check{\mathfrak{g}}$. Moreover, $\nabla_{\check{G}}$ is the pushout of $\nabla_{\check{G}'}$, i.e., the diagram*

$$\begin{array}{ccc} \mathrm{Rep}(\check{G}) & \xrightarrow{\text{restriction}} & \mathrm{Rep}(\check{G}') \\ & {\scriptstyle\nabla_{\check{G}}} \searrow \quad \swarrow {\scriptstyle\nabla_{\check{G}'}} & \\ & \mathrm{Conn}(\mathbb{G}_{m,t}) & \end{array} \tag{2.2.2}$$

is commutative.

Example 2.3. 1. When $\check{G}'$ and $\check{G}$ are of the same type, they have the same Lie algebra. We have the 'trivial functoriality' in this case. In other words, we have $\check{\mathfrak{g}} = \check{\mathfrak{g}}'$ and take $N = N', E = E'$. Then for each representation V of $\check{G}$, we have $\nabla_{\check{G}}(V) = \nabla_{\check{G}'}(V)$.

2. When $\check{G}' = \mathrm{SO}_{2n+1}$ and $\check{G} = \mathrm{SO}_{2n+2}$, since the standard representation of SO_{2n+2} restricted as a representation of SO_{2n+1} splits into a direct sum of the standard representation of SO_{2n+1} and a trivial representation, we have

$$\nabla_{\mathrm{SO}_{2n+2}}(\mathbb{C}^{2n+2}) = \nabla_{\mathrm{SO}_{2n+1}}(\mathbb{C}^{2n+1}) \oplus \mathcal{O}_{\mathbb{G}_m}. \tag{2.2.3}$$

3. When $\check{G}' = G_2$ and $\check{G} = \mathrm{SO}_7$, the standard representation $V = \mathbb{C}^7$ of SO_7 remains irreducible when viewed as a representation of G_2. In this case, we have

$$\nabla_{G_2}(V) = \nabla_{\mathrm{SO}_7}(V), \tag{2.2.4}$$

which corresponds to the hypergeometric equation in Example 2.1.

4. When $\check{G}' = F_4$ and $\check{G} = E_6$, we have a decomposition $\mathfrak{e}_6 = \mathfrak{f}_4 \oplus \tilde{V}$, where $\tilde{V}$ is the 26-dimensional representation of F_4. Note that $\dim(E_6) = 78$ and $\dim(F_4) = 52$. Hence, the (nontrivial) 27-dimensional minuscule E_6-representation V restricts to a nontrivial representation of F_4, which therefore has to be $\mathbb{C} \oplus \tilde{V}$ (*cf.* [Ada96, Lem. 14.4]). By the functoriality (2.2.2), we have

$$\nabla_{E_6}(V) = \nabla_{F_4}(\widetilde{V}) \oplus \mathcal{O}_{\mathbb{G}_m} \tag{2.2.5}$$

for compatible choices of N and E on both sides.

$\diamondsuit$

2.3 Heinloth–Ngo–Yun's Kloosterman connections

Heinloth–Ngô–Yun constructed (ℓ-adic) Kloosterman sheaves for reductive groups in [HNY13]. Applying their construction to $\mathscr{D}$-modules, we also get some $\check{G}$-connections on $\mathbb{G}_m$.

For each almost simple group G, we fix a maximal torus and a Borel subgroup $T \subset B \subset G$. Recall that the *loop group* LG (resp. the *positive loop group* L^+G) is the fppf sheaf on the category of k-algebras, defined by

$$R \mapsto G(R((t))) \quad (\text{resp. } R \mapsto G(R[[t]])).$$

The *affine Grassmannian* $\mathrm{Gr} = \mathrm{Gr}_G$ is the fppf-quotient LG/L^+G [Zhu17b, Prop. 1.3.6], which is represented by an ind-scheme.

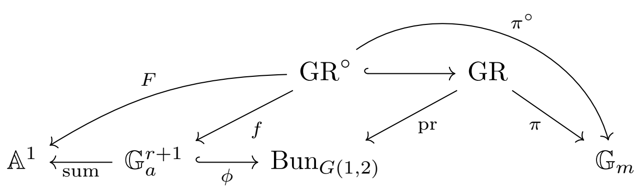
The group L^+G has a natural action on Gr, and the orbits are indexed by dominant cocharacters $\lambda^\vee \in \Phi^{\vee +}$. This orbit and its closure are denoted by $\mathrm{Gr}_{\lambda^\vee}$ and $\mathrm{Gr}_{\leq \lambda^\vee}$ respectively [Zhu17b, Prop. 2.1.5]. We denote by $\mathrm{IC}_{\lambda^\vee}$ the intersection complex on $\mathrm{Gr}_{\leq\lambda^\vee}$. For each irreducible representation V of the dual group $\check{G}$ of highest weight $\lambda^\vee$, we also write $\mathrm{IC}_V = \mathrm{IC}_{\lambda^\vee}$.

The Beilinson–Drinfeld Grassmanian, $\pi\colon \mathrm{GR} \to \mathbb{G}_m$, over $\mathbb{G}_m$ is a global version of affine Grassmannians, see for example [Zhu17b, Def. 3.11]. We denote by $\mathbb{G}_m^{\mathrm{rot}}$ the torus $\mathbb{G}_m$ that acts on $\mathbb{G}_m$ by multiplication. The specified action of $\mathbb{G}_m^{\mathrm{rot}}$ on $\mathbb{G}_m$ induces an action of $\mathbb{G}_m^{\mathrm{rot}}$ on GR such that π is $\mathbb{G}_m^{\mathrm{rot}}$-equivariant. Using this action, we can trivialize the fibration π as

$$\mathrm{GR} \simeq \mathrm{Gr} \times \mathbb{G}_m \to \mathbb{G}_m,$$

where Gr is the affine Grassmannian for G, see [HNY13, (5.11)].

Let $\mathrm{Bun}_{G(1,2)}$ be the moduli stack of G-bundles on $\mathbb{G}_m$ with some level structures at 0 and ∞ determined by level groups $I(1)$ and $I(2)$, as defined in [HNY13, §1.2]. Then there is a projection $\mathrm{pr}\colon \mathrm{GR} \to \mathrm{Bun}_{G(1,2)}$, and we have the following diagram in [HNY13, §5.2] or [Yun15, (2.2)] as follows:



(2.3.1)

where $\phi\colon \mathbb{G}_a^{r+1} \simeq T \times I(1)/I(2) \hookrightarrow \mathrm{Bun}_{G(1,2)}$ is the inclusion of the big open cell [HNY13, Cor. 1.3 (4)], the open sub-ind-scheme $\mathrm{GR}^\circ \subset \mathrm{GR}$ is the inverse image of the big cell.

Similar to the Beilinson–Drinfeld Grassmanian, we have $\mathrm{GR}^\circ \simeq \mathrm{Gr}^\circ \times \mathbb{G}_m$. Let $\pi^\circ \colon \mathrm{GR}^\circ \to \mathbb{G}_m$ and $\pi_{\mathrm{Gr}} \colon \mathrm{GR} \to \mathrm{Gr}$ be the corresponding projections, and we write $\mathrm{IC}_{V,\mathrm{GR}^\circ} := \pi_{\mathrm{Gr}}^{+}\mathrm{IC}_V|_{\mathrm{Gr}^\circ}$. Then the two isomorphic complexes of $\mathscr{D}$-modules

$$\pi^\circ_{\dagger}(\mathcal{E}^F \otimes \mathrm{IC}_{V,\mathrm{GR}^\circ}) \simeq \pi^\circ_{+}(\mathcal{E}^F \otimes \mathrm{IC}_{V,\mathrm{GR}^\circ}) \tag{2.3.2}$$

are connections over $\mathbb{G}_m$, denoted by $\mathrm{Kl}_{\check{G}}(V)$, where $\mathcal{E}^F = (\mathcal{O}_{\mathrm{GR}^\circ}, \mathrm{d} + \mathrm{d}F)$ is the exponential $\mathscr{D}$-module. Furthermore, they can be upgraded to a tensor functor

$$\mathrm{Kl}_{\check{G}} \colon \mathrm{Rep}(\check{G}) \to \mathrm{Conn}(\mathbb{G}_m),$$

see [HNY13, Thm. 1(1)] or [Yun15, Thm. 2.2.1].

Heinloth–Ngô–Yun conjectured that these $\mathscr{D}$-modules should coincide with Frenkel–Gross's connections [HNY13, Conj. 2.16], and this conjecture has been proven in [Zhu17a, §6]:

Theorem 2.4 ([Zhu17a])**.** *The tensor functor $\mathrm{Kl}_{\check{G}}$ is isomorphic to the Frenkel–Gross connection $\nabla_{\check{G}}$.*

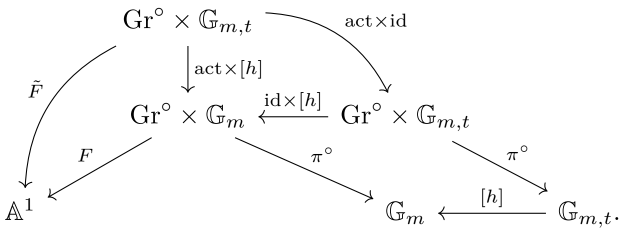
2.4 Homogeneity properties

Let G be an almost simple group of adjoint type, so the half sum of positive coroots $\rho^\vee$ is a cocharacter, and recall that we have fixed a maximal torus $T \subset G$. In [Yun15, §2.6.1], Yun constructed an ind-scheme $\mathfrak{S} = \mathfrak{S}_1 \times \mathbb{G}_m$ isomorphic to $\mathrm{GR}^\circ = \mathrm{Gr}^\circ \times \mathbb{G}_m$, compatible with the $\mathbb{G}_m^{\mathrm{rot}}$-action. In [Yun15, §2.6.4], he also defined an embedding of 1-dimensional subtorus $\mathbb{G}_m$ of $T \times \mathbb{G}_m^{\mathrm{rot}}$, labeled $\mathbb{G}_m^{(\rho^\vee,h)}$, by

$$\mathbb{G}_m^{(\rho^\vee,h)} \ni t \mapsto (\rho^\vee(t), t^h) \in T \times \mathbb{G}_m^{\mathrm{rot}}.$$

Hence, the torus $\mathbb{G}_m^{(\rho^\vee,h)}$ inherits an action on $\mathrm{GR}^\circ = \mathrm{Gr}^\circ \times \mathbb{G}_m$. Moreover, F is $\mathbb{G}_m^{(\rho^\vee,h)}$-equivariant and $\mathbb{G}_m^{(\rho^\vee,h)}$ acts on both $\mathbb{G}_a^{r+1}$ and $\mathbb{A}^1$ by the multiplication on each factor.

Recall that $[h] \colon \mathbb{G}_{m,t} \to \mathbb{G}_{m,t}$ denotes the h-th power morphism, $\mathrm{act} \colon \mathrm{Gr}^\circ \times \mathbb{G}_m \to \mathrm{Gr}^\circ$ the action of $\mathbb{G}_m^{(\rho^\vee,h)}$ on Gr°, and $\tilde{F} = F \circ (\mathrm{act} \times [h])$. These morphisms are illustrated in the following diagram:



(2.4.1)

Then we deduce that

$$\begin{aligned} \tilde{F}(x,t) &= F(\mathrm{act}(x,t), t^h) = && \text{definition of } \tilde{F} \\ &= F(t \cdot (x,1)) && \text{definition of the action of } \mathbb{G}_m^{(\rho^\vee,h)} \\ &= tF(x,1) && F \text{ is } \mathbb{G}_m^{(\rho^\vee,h)}\text{-equivariant.} \end{aligned}$$

Notice that $\pi^\circ \circ (\mathrm{act} \times \mathrm{id}) = \pi^\circ$, we have

$$\begin{aligned} [h]^+\mathrm{Kl}_{\check{G}}(V) &= [h]^+\pi^\circ_+(\mathcal{E}^F \otimes \mathrm{IC}_{V,\mathrm{GR}^\circ}) \\ &\cong \pi^\circ_+(\mathcal{E}^{F\circ(\mathrm{id}\times[h])} \otimes \mathrm{IC}_{V,\mathrm{GR}^\circ}) \\ &\cong \pi^\circ_+(\mathrm{act}\times\mathrm{id})_+(\mathrm{act}\times\mathrm{id})^+(\mathcal{E}^{F\circ(\mathrm{id}\times[h])} \otimes \mathrm{IC}_{V,\mathrm{GR}^\circ}) \\ &\cong \pi^\circ_+(\mathcal{E}^{\tilde{F}} \otimes \mathrm{IC}_{V,\mathrm{GR}^\circ}) \\ &\cong \pi^\circ_+(\mathcal{E}^{t\cdot F(x,1)} \otimes \mathrm{IC}_{V,\mathrm{GR}^\circ}), \end{aligned} \tag{2.4.2}$$

where in the third isomorphism, we used the fact that $\mathrm{act} \times \mathrm{id}$ is an isomorphism.

Combined with Theorem 2.4, we have the following proposition:

Proposition 2.5. The connection $\widetilde{\nabla}_{\check{G}}(V) = [h]^{+}\nabla_{\check{G}}(V)$ is of the form

$$\pi^{\circ}_{+}(\mathcal{E}^{t\cdot g} \otimes \mathrm{IC}_{V,\mathrm{GR}^{\circ}}),$$

where $g = F|_{\mathrm{Gr}^{\circ}\times\{1\}} \colon \mathrm{Gr}^{\circ} \to \mathbb{A}^1 : x \mapsto F(x,1)$, $t\cdot g$ is viewed as a regular function on $\mathrm{Gr}^{\circ} \times \mathbb{G}_{m,t}$, and π° is the projection from $\mathrm{Gr}^{\circ} \times \mathbb{G}_{m,t}$ to $\mathbb{G}_{m,t}$.

3 The irregular Hodge numbers of Frenkel–Gross connections

3.1 Recollections on rescalable integrable mixed twistor $\mathscr{D}$-modules

Recall from the introduction that we have for any smooth complex algebraic variety M the category $\mathrm{MHM}(M,\mathbb{R})$ of *mixed Hodge modules* on M with coefficients in $\mathbb{R}$, which is abelian and which is the category of $\mathbb{R}$-polarized mixed Hodge structures if M is a point. It comes with a forgetful functor, sending a mixed Hodge module to its underlying regular holonomic $\mathscr{D}_M$-module. Furthermore, this forgetful functor induces a forgetful functor on the bounded derived categories $D^b(\mathrm{MHM}(M),\mathbb{R}) \to D^b_h(\mathscr{D}_M)$, which is compatible with the six functor formalism (in the sense explained in the introduction).

There is a related category $\mathrm{MTM}^{\mathrm{int}}(M,\mathbb{R})$ of *integrable mixed twistor $\mathscr{D}_M$-modules*, which can be regarded as an enhancement of $\mathrm{MHM}(M)$. It is built in a parallel way (with major new challenges though) to the theory of mixed Hodge modules but uses $\mathscr{R}^{\mathrm{int}}_M = \mathscr{D}_M[z]\langle z^2\partial_z\rangle$-modules instead of $\mathscr{D}_M$-modules. The underlying $\mathscr{R}$-modules $\mathscr{M}$ of objects in $\mathrm{MTM}^{\mathrm{int}}(M)$ give rise to (not necessarily regular) holonomic $\mathscr{D}_M$-modules through the functor $\Xi_{\mathrm{dR}}(\mathscr{M}) = \mathscr{M}/(z-1)\mathscr{M}$. The induced functor $\Xi_{\mathrm{dR}} \colon D^b\mathrm{MTM}^{\mathrm{int}}(M) \to D^b_h(\mathscr{D}_M)$ is also compatible with the six functor formalism and nearby cycle functors [Moc15, Paragraph after Prop. 14.1.24]. Furthermore, there is a fully faithful exact functor

$$v \colon D^b(\mathrm{MHM}(M)) \to D^b\mathrm{MTM}^{\mathrm{int}}(M) \tag{3.1.1}$$

that is built via the Rees construction and which is compatible with the six functor formalism, as detailed in [Moc15, Prop. 13.5.5 & Prop. 14.3.29].

We are interested in a category of "irregular Hodge modules" that lies in between $\mathrm{MHM}(M,\mathbb{R})$ and $\mathrm{MTM}^{\mathrm{int}}(M)$. It was first defined and studied by Sabbah ([Sab18a]). We will rather work with two related categories (studied by Mochizuki in [Moc21]), namely, the category of rescalable integrable twistor $\mathscr{D}$-modules and the category of exponential mixed Hodge modules.

Definition 3.1. The category $\mathrm{EMHM}(M)$ of *exponential mixed Hodge modules* on M is defined as the full subcategory of $\mathrm{MHM}(\mathbb{A}^1_\theta \times M,\mathbb{R})$ whose objects N^{H} satisfies $\pi_{M*}N^{\mathrm{H}} = 0$ for the projection $\pi_M \colon \mathbb{A}^1_\theta \times M \to M$. When M is a point, we recover the category of exponential mixed Hodge structures, as defined in [KS11]. ♢

The category $\mathrm{EMHM}(M)$ can also be defined as the essential image of a projector $P_* \colon \mathrm{MHM}(\mathbb{A}^1_\theta \times M,\mathbb{R}) \to \mathrm{MHM}(\mathbb{A}^1_\theta \times M,\mathbb{R})$, which is defined in [Moc21, (155)], see also the discussion right after it.

Now we explain how to embed $\mathrm{EMHM}(M)$ as a subcategory of $\mathrm{MTM}^{\mathrm{int}}(M)$ and to attach an irregular Hodge filtration to its objects. Mochizuki constructed a functor

$$u \colon \mathrm{MTM}^{\mathrm{int}}(\mathbb{A}^1_\tau \times M) \to \mathrm{MTM}^{\mathrm{int}}(M) \tag{3.1.2}$$

in [Moc21, §11.4.4] by sending $\mathscr{M}$ to an object $u(\mathscr{M})$ such that $\iota_! u(\mathscr{M}) = \psi^{(1)}_{\tau^{-1}}\mathscr{M}$.

Definition 3.2 ([Moc21, Def. 11.38])**.** The essential image of u is a full subcategory of $\mathrm{MTM}^{\mathrm{int}}(M)$, called *rescalable mixed twistor $\mathscr{D}_Y$-modules* and denoted by $\mathrm{MTM}^{\mathrm{int}}_{\mathrm{resc}}(M)$. ♢

Let $\mathbf{FT}_Y \colon \mathrm{MTM}^{\mathrm{int}}(\mathbb{A}^1_\theta \times M) \to \mathrm{MTM}^{\mathrm{int}}(\mathbb{A}^1_\tau \times M)$ be the partial Fourier transform [Moc21, §10.2] of integrable twistor modules relative to M where τ denotes the coordinate of the affine line dual to $\mathbb{A}^1_\theta$. Then Mochizuki proved the following theorem:

Theorem 3.3 ([Moc21, Thm. 11.45])**.** *The functor*

$$B \colon \mathrm{MHM}(\mathbb{A}^1_\theta \times M,\mathbb{R}) \xrightarrow{u\circ\mathbf{FT}_Y\circ v} \mathrm{MTM}^{\mathrm{int}}(M), \tag{3.1.3}$$

induces an equivalence of categories of $B_* \colon \mathrm{EMHM}(M) \xrightarrow{\sim} \mathrm{MTM}^{\mathrm{int}}_{\mathrm{resc}}(M)$.

Moreover, Mochizuki showed that the functor B is compatible with the six functors formalism, nearby and vanishing cycle functors, and duality in [Moc21, Props. 11.46-11.48]. In particular, these operators preserve rescalable integrable mixed twistor $\mathscr{D}$-modules.

As explained above, the *de Rham* functor $\Xi_{\mathrm{dR}} : \mathrm{MTM}^{\mathrm{int}}_{\mathrm{res}}(M) \subset \mathrm{MTM}(M) \to \mathrm{Mod}_h(\mathscr{D}_Y)$ sends a rescalable integrable twistor $\mathscr{D}$-module $\mathcal{M}$ to the holonomic $\mathscr{D}$-module $\mathcal{M}/(z-1)\mathcal{M}$. For any $\mathcal{T} \in \mathrm{MTM}^{\mathrm{int}}_{\mathrm{res}}(M)$ one can then equip $\Xi_{\mathrm{dR}}(\mathcal{T})$ with an irregular Hodge filtration $F^{\bullet}_{\mathrm{irr}}$, see [Sab18a, Thm. 0.3] and [Moc21, Cor. 1.6]. In particular, we denote by h^p_{irr} the irregular Hodge numbers, i.e. the (generic) ranks of the graded pieces of $F^{\bullet}_{\mathrm{irr}}\Xi_{\mathrm{dR}}(\mathcal{T})$.

To end this section, we collect a unicity result, which will be needed later.

Lemma 3.4. Let M be a smooth quasi-projective variety, and let N be a semi-simple holonomic $\mathscr{D}_Y$-module. Then there is up to isomorphism at most one object $\mathcal{T} \in \mathrm{MTM}^{\mathrm{int}}_{\mathrm{resc}}(M)$ of weight 0 such that $\Xi_{\mathrm{dR}}(\mathcal{T}) \cong N$.

Proof. We choose a compactification $j_M : M \hookrightarrow \overline{M}$, where $\overline{M}$ is smooth and projective. Then the intermediate extension $j_{M,\dagger+}N$ is a semi-simple holonomic $\mathscr{D}_{\overline{M}}$-module. Now suppose that we have $\mathcal{T}_1, \mathcal{T}_2 \in \mathrm{MTM}^{\mathrm{int}}_{\mathrm{resc}}(M)$ such that $\Xi_{\mathrm{dR}}(\mathcal{T}_1) \cong \Xi_{\mathrm{dR}}(\mathcal{T}_2)$. According to [Moc15, Thm. 14.3.16 & Prop. 14.3.17], for $i \in \{1,2\}$, we have the objects $j_{M,*}\mathcal{T}_i$ and $j_{M,!}\mathcal{T}_i$ in $\mathrm{MTM}(\overline{M})$ and by loc.cit., Proposition 14.3.18, there is a morphism $j_{M,!}\mathcal{T}_i \to j_{M,*}\mathcal{T}_i$. Denote by $j_{M,!*}\mathcal{T}_i$ the respective images in $\mathrm{MTM}(\overline{M})$. Since taking the de Rham functor Ξ_{dR} commutes with taking direct images (with and without proper support) of integrable mixed twistor modules resp. of holonomic $\mathscr{D}$-modules, we see that $\Xi_{\mathrm{dR}}(j_{M,!*}\mathcal{T}_i) = j_{M,\dagger+}N$ for both $i = 1, 2$. Now by [Moc11, Thm. 1.4.4], there is a unique (wild and pure) analytic twistor module $\overline{\mathcal{T}}$ on $\overline{M}$ such that $\Xi_{\mathrm{dR}}(\overline{\mathcal{T}}) = (j_{M,\dagger+}N)^{an}$. Hence, we obtain an isomorphism

$$\phi : (j_{M,!*}\mathcal{T}_1)^{an} \longrightarrow (j_{M,!*}\mathcal{T}_2)^{an},$$

but since $\overline{M}$ is projective, the categories $\mathrm{MTM}(M)^{an}$ and $\mathrm{MTM}(\overline{M})$ are equivalent (this follows, e.g., from [Moc15, Lem. 14.1.2]). Therefore we obtain an isomorphism

$$\phi^{\mathrm{alg}} : j_{M,!*}\mathcal{T}_1 \longrightarrow j_{M,!*}\mathcal{T}_2,$$

which restricts to an isomorphism

$$\phi^{\mathrm{alg}}_{|M} : \mathcal{T}_1 \longrightarrow \mathcal{T}_2,$$

of (algebraic) mixed twistor modules on M. If we now consider $\mathcal{T}_i$ as integrable twistor modules on M, then by [Sab18a, Rem. 1.39], $\phi^{\mathrm{alg}}_{|M}$ is also an isomorphism in $\mathrm{MTM}^{\mathrm{int}}(M)$. Now since $\mathrm{MTM}^{\mathrm{int}}_{\mathrm{res}}(M)$ is a a full subcategory of $\mathrm{MTM}^{\mathrm{int}}(M)$, $\phi_{|M}$ is an isomorphism of (algebraic) rescalable integrable mixed twistor $\mathscr{D}$-modules, as required. ☐

3.2 Rescalable integrable mixed twistor $\mathscr{D}$-modules attached to $\nabla_{\check{G}}(V)$.

In this section, we construct objects in $\mathrm{MTM}^{\mathrm{int}}_{\mathrm{res}}(\mathbb{G}_{m,t})$, which will be sent to Frenkel–Gross connections under the de Rham functors.

For a regular function $f : U \to \mathbb{A}^1$, there exists an integrable twistor $\mathscr{D}$-modules $\mathcal{T}^{f/z} \in \mathrm{MTM}^{\mathrm{int}}(U)$, which is sent to the exponential $\mathscr{D}_U$-module $\mathcal{E}^f = (\mathcal{O}_U, \mathrm{d} + \mathrm{d}f)$ under the de Rham functor. See [Sab18a, Discussion before Thm. 0.2] for details. One can verify that it is rescalable, as it is the image of $\mathcal{T}^{\tau \cdot f/z}$ under the functor u mentioned above in (3.1.2) (notice that $\mathcal{T}^{\tau \cdot f/z}$ is a smooth $\mathcal{R}_{\mathbb{A}^1_\tau \times U}$-module, therefore, $\psi^{(1)}_{\tau-1}\mathcal{T}^{\tau \cdot f/z} = \mathcal{T}^{f/z}$).

The case of $\nabla_{\check{G}}(V)$ Let V be an irreducible $\check{G}$-representation of the highest weight $\lambda^\vee$, then $\mathrm{Gr}_{\leq \lambda^\vee}$ has dimension $d_V := \langle 2\rho, \lambda^\vee \rangle$. Then the mixed Hodge module $\mathrm{IC}^{\mathrm{H}}_{V,\mathrm{GR}^\circ}(\frac{d_V+1}{2})$ has weight 0. Taking notation from (2.3.1), we define

$$\nabla^{\mathrm{H}}_{\check{G}}(V) := \pi^\circ_*\Big(\mathcal{T}^{F/z} \otimes v(\mathrm{IC}_{V,\mathrm{GR}^\circ})\left(\tfrac{d_v+1}{2}\right)\Big) \tag{3.2.1}$$

where v is the Rees functor (3.1.1). By the compatibility of the six functor formalism with the de Rham functor Ξ_{dR} we conclude from (2.3.2)

$$\Xi_{\mathrm{dR}}\left(\nabla^{\mathrm{H}}_{\check{G}}(V)\right) = \pi^\circ_+(\mathcal{E}^F \otimes \mathrm{IC}_{V,\mathrm{GR}^\circ})|_{\mathbb{G}_{m,t}} = \nabla_{\check{G}}(V).$$

Hence, we call $\nabla^{\mathrm{H}}_{\check{G}}(V)$ the rescalable integrable twistor $\mathscr{D}$-module attached to $\nabla_{\check{G}}(V)$.

Moreover, in (2.3.2), the forget support morphism $\pi^{\circ}_{\dagger}(\mathcal{E}^F \otimes \mathrm{IC}_{V,\mathrm{GR}^\circ})|_{\mathbb{G}_{m,t}} \to \pi^{\circ}_{+}(\mathcal{E}^F \otimes \mathrm{IC}_{V,\mathrm{GR}^\circ})|_{\mathbb{G}_{m,t}}$ is an isomorphism. As Ξ_{dR} is an exact functor, the forget support morphism

$$\pi^{\circ}_{!}\Big(\mathcal{T}^{F/z} \otimes v(\mathrm{IC}_{V,\mathrm{GR}^\circ})\Big)|_{\mathbb{G}_{m,t}} \to \pi^{\circ}_{*}\Big(\mathcal{T}^{F/z} \otimes v(\mathrm{IC}_{V,\mathrm{GR}^\circ})\Big)|_{\mathbb{G}_{m,t}}$$

is also an isomorphism. Hence, by the formalism of weights, $\nabla^{\mathrm{H}}_{\check{G}}(V)$ is a pure twistor module of weight 0.

The case of $\widetilde{\nabla}_{\check{G}}(V)$ Let $[h]\colon \mathbb{G}_{m,t} \to \mathbb{G}_{m,t}$ be the h-th power map. We define

$$\widetilde{\nabla}^{\mathrm{H}}_{\check{G}}(V) := [h]^*\nabla^{\mathrm{H}}_{\check{G}}(V) \in \mathrm{MTM}^{\mathrm{int}}_{\mathrm{resc}}(\mathbb{G}_{m,t}). \tag{3.2.2}$$

By the compatibility of the six functor formalism with the de Rham functor Ξ_{dR} we conclude that

$$\Xi_{\mathrm{dR}}\Big(\widetilde{\nabla}^{\mathrm{H}}_{\check{G}}(V)\Big) = [h]^+\nabla_{\check{G}}(V) = \widetilde{\nabla}_{\check{G}}(V).$$

Hence, we call $\widetilde{\nabla}^{\mathrm{H}}_{\check{G}}(V)$ the rescalable integrable twistor $\mathscr{D}$-module attached to $\widetilde{\nabla}_{\check{G}}(V)$.

When $\check{G}$ is simply-connected (i.e. G is adjoint), we have by homogeneity (see Proposition 2.5) that

$$\widetilde{\nabla}^{\mathrm{H}}_{\check{G}}(V) = \pi^{\circ}_{*}\left(\mathcal{T}^{(t\cdot g)/z} \otimes v(\mathrm{IC}_{V,\mathrm{GR}^\circ})\left(\tfrac{d_V+1}{2}\right)\right)|_{\mathbb{G}_{m,t}}. \tag{3.2.3}$$

The case of $\widetilde{\nabla}_{\check{G}}(V) \otimes \mathbb{L}_\chi$ For a rational number $a \in \mathbb{Q}\backslash\mathbb{Z}$, let $\chi = \exp(2\pi i a) \neq 1$ and let $\mathbb{L}_\chi$ be the connection defined by $\mathbb{L}_\chi := (\mathcal{O}_{\mathbb{G}_{m,t}}, \mathrm{d} + a\frac{\mathrm{d}t}{t})$ (sometimes called Kummer module). We then have the $\mathscr{D}_{\mathbb{A}^1_t}$-module $\widetilde{\nabla}_{\check{G}}(V)_\chi := j_+(\widetilde{\nabla}_{\check{G}}(V) \otimes \mathbb{L}_\chi)$. On the other hand, $\mathbb{L}_\chi$ underlies a pure complex Hodge module of rank 1 that we denote by $\mathbb{L}^{\mathrm{H}}_\chi$.

We then consider the rescalable integrable mixed twistor $\mathscr{D}$-module

$$\widetilde{\nabla}^{\mathrm{H}}_{\check{G}}(V)_\chi := j_*\widetilde{\nabla}^{\mathrm{H}}_{\check{G}}(V) \otimes j_*v(\mathbb{L}^{\mathrm{H}}_\chi) \in \mathrm{MTM}^{\mathrm{int}}_{\mathrm{resc}}(\mathbb{A}^1_t) \tag{3.2.4}$$

that is attached to $\widetilde{\nabla}_{\check{G}}(V)_\chi$ in the sense that $\Xi_{\mathrm{dR}}(\widetilde{\nabla}^{\mathrm{H}}_{\check{G}}(V)_\chi) = \widetilde{\nabla}_{\check{G}}(V)_\chi$.

Functoriality Notice that $\nabla^{\mathrm{H}}_{\check{G}}(V) = \pi^{\circ}_{*}(\mathrm{IC}^{\mathrm{H}}_{V,\mathrm{GR}^\circ} \otimes \mathcal{T}^{F/z})$ can be seen as Hecke-eigen twistor $\mathscr{D}$-modules of the Hecke operator Hk_V in [HNY13, §2.3], i.e., by replacing the Artin–Schreier sheaf $\mathscr{L}_{\psi(x)}$ with $\mathcal{T}^{x/z}$ in the definition of $A_{\phi,\chi}$ in [HNY13, §2.2] and then applying the Hecke operator in the sense of twistor $\mathscr{D}$-modules to $A_{\phi,\chi}$, we recover $\nabla^{\mathrm{H}}_{\check{G}}(V)$ are Hecke-eigen twistor $\mathscr{D}$-modules. As we can compose the Hecke operators, and have $\mathrm{Hk}_V \circ \mathrm{Hk}_W = \mathrm{Hk}_{V\otimes W}$ for any two representations of $\check{G}$, we deduce that $\nabla^{\mathrm{H}}_{\check{G}}$ can be upgraded to a tensor functor

$$\nabla^{\mathrm{H}}_{\check{G}}\colon \mathrm{Rep}_{\check{G}} \to \mathrm{MTM}^{\mathrm{int}}_{\mathrm{resc}}(\mathbb{G}_{m,t}). \tag{3.2.5}$$

As explained in Section 3.2, each object in the category of rescalable mixed twistor modules is equipped with an irregular Hodge filtration in a functorial way. When restricted to smooth objects (that is, mixed twistor $\mathscr{D}$-modules whose image under the de Rham functor are smooth $\mathscr{D}$-module, i.e. connections), this gives a tensor functor

$$\mathrm{MTM}^{\mathrm{int}}_{\mathrm{sm,resc}}(\mathbb{G}_{m,t}) \xrightarrow{\mathrm{IrrFil}} \mathrm{Fil}_{\mathbb{G}_{m,t}}$$

to the category of vector bundles on $\mathbb{G}_{m,t}$ filtered by subbundles. Since the essential image of ∇^{H} lies in $\mathrm{MTM}^{\mathrm{int}}_{\mathrm{sm,resc}}(\mathbb{G}_{m,t})$, we can consider the composition

$$\mathrm{Rep}(\check{G}) \xrightarrow{\nabla^{\mathrm{H}}} \mathrm{MTM}^{\mathrm{int}}_{\mathrm{sm,resc}}(\mathbb{G}_{m,t}) \xrightarrow{\mathrm{IrrFil}} \mathrm{Fil}_{\mathbb{G}_{m,t}}.$$

Similar to [Lov17, §3.2.4], we call a functor $F\colon \mathrm{Rep}(\check{G}) \to \mathrm{Fil}_{\mathbb{G}_m}$ an *η/k-filtration* for some dominant cocharacter η and some integer k, if for each representation V of $\check{G}$, the decreasing filtration by subbundles $F(V)$ is given by

$$F^\alpha\nabla_{\check{G}}(V) := \mathcal{O}_{\mathbb{G}_m} \otimes_{\mathbb{C}} \left(\bigoplus\nolimits_{i\geq k\cdot\alpha} V_i\right),$$

where $V = \bigoplus_{i\in\mathbb{Z}} V_i$ is the decomposition of V induced by the $\mathbb{G}_m$-action $\eta_V : \mathbb{G}_m \xrightarrow{\eta} \check{G} \to \mathrm{GL}(V)$.

Definition 3.5. We say that the irregular Hodge filtrations of $\nabla^{\mathrm{H}}_{\check{G}}$ are determined by η/k for some dominant cocharacter $\eta\colon \mathbb{G}_m \to \check{G}$ and a positive integer k if the composition $\mathrm{IrrFil} \circ \nabla^{\mathrm{H}}$ is an η/k-filtration. We say that the irregular Hodge numbers of $\nabla^{\mathrm{H}}_{\check{G}}$ are determined by η/k if for any $V \in \mathrm{Rep}(\check{G})$ the ranks of $\mathrm{gr}^{\alpha}(\mathrm{IrrFil} \circ \nabla^{\mathrm{H}}(V))$ equal the dimensions of the eigenspaces $V_{k\cdot\alpha}$ of the operator $\eta(V)$. ♢

Clearly, if the irregular Hodge filtration of $\nabla^{\mathrm{H}}_{\check{G}}$ is determined by η/k, then so are its irregular Hodge numbers. The converse is, of course, not true in general (see also Conjecture 6.2 below).

Using these notions, we can summarize the above discussion as follows.

Proposition 3.6. The Frenkel–Gross connection $\nabla_{\check{G}}$ underlies a unique rescalable integrable mixed twistor $\mathscr{D}$-module $\nabla^{\mathrm{H}}_{\check{G}}$ of weight 0, i.e., a tensor functor

$$\nabla^{\mathrm{H}}_{\check{G}}\colon \mathrm{Rep}_{\check{G}} \to \mathrm{MTM}^{\mathrm{int}}_{\mathrm{resc}}(\mathbb{G}_{m,t}).$$

In particular, there is a uniquely defined irregular Hodge filtration on $\nabla_{\check{G}}$, and it is determined by η/k for some cocharacter η and some integer k.

Proof. We already constructed rescalable integrable mixed twistor $\mathscr{D}$-module $\nabla^{\mathrm{H}}_{\check{G}}$ on $\nabla_{\check{G}}$ in (3.2.5). To show the uniqueness of $\nabla^{\mathrm{H}}_{\check{G}}$, we decompose any representation W of $\check{G}$ as $W = \bigoplus W_i \otimes M_i$ with W_i irreducible and M_i the multiplicities. By the functoriality of $\nabla^{\mathrm{H}}_{\check{G}}$, we have

$$\nabla^{\mathrm{H}}_{\check{G}}(W) = \bigoplus \nabla^{\mathrm{H}}_{\check{G}}(W_i) \otimes M_i^{\mathrm{H}},$$

with M_i^{H} being trivial. As in Lemma 3.4, there is at most one rescalable integrable mixed twistor $\mathscr{D}$-modules of weight 0 on an irreducible connection on $\mathbb{G}_m$. So each $\nabla^{\mathrm{H}}_{\check{G}}(W_i)$ is unique. Therefore, $\nabla^{\mathrm{H}}_{\check{G}}(W)$ is also unique.

At last, since each $\nabla_{\check{G}}(V)$ is a connection on $\mathbb{G}_{m,t}$, its associated rescalable integrable mixed twistor $\mathscr{D}$-module gives rise to an irregular Hodge filtration (by subbundles) in a functorial way. In other words, we have a tensor functor from $\mathrm{Rep}_{\check{G}}$ to ($\frac{1}{k}\mathbb{Z}$-indexed) filtered vector bundles on $\mathbb{G}_{m,t}$ for some integer k. By [Lov17, Lem. 3.3.1], this functor is then necessarily an η/k-filtration for some cocharacter η. □

3.3 Inverse Fourier transforms of $\widetilde{\nabla}_{\check{G}}(V)$

In this section, we study the (inverse) Fourier transform of the Frenkel-Gross connection (resp. of its variant $\widetilde{\nabla}_{\check{G}}$ or some twisted version thereof). From (2.3.2) we will deduce that when using a non-trivial twisting, they underly pure polarizable Hodge modules. This property will be pivotal to our computations of Hodge numbers in the next section.

We first work in a slightly more general geometric framework to simplify our proof. The situation will be specified in the case of Frenkel-Gross connections at the end of this section.

The setup Let Y be a quasi-projective variety over $\mathbb{C}$ and g a regular function on Y. We consider the product $Y \times \mathbb{A}^1_t$ and we denote by pr_t resp. by pr_Y the projection $Y \times \mathbb{A}^1_t \to \mathbb{A}^1_t$ resp. the projection $Y \times \mathbb{A}^1_t \to Y$. We have the following diagram:

$$\mathbb{A}^1 \xleftarrow{\;t\cdot g\;} Y \times \mathbb{A}^1_t \xrightarrow{\;\mathrm{pr}_t\;} \mathbb{A}^1_t.$$

We start by collecting a few general facts about Fourier transforms of (complexes of) $\mathscr{D}_{\mathbb{A}^1_t}$-modules that we will use later. Recall from the introduction that the Fourier transform of a an object $\mathcal{M}^\bullet \in D^b_h(\mathbb{A}^1_t)$ denoted by $\mathrm{FT}^{\pm 1}(\mathcal{M}^\bullet)$, is the complex of holonomic $\mathscr{D}_{\mathbb{A}^1_s}$-modules defined by

$$\mathrm{pr}_{s+}(\mathrm{pr}_t^+ \mathcal{M}^\bullet \otimes \mathcal{E}^{\pm t\cdot s}),$$

where pr_t and pr_s are the projections from $\mathbb{A}^1_t \times \mathbb{A}^1_s$ to $\mathbb{A}^1_t$ and $\mathbb{A}^1_s$ respectively. Similarly, we can define the Fourier transform of $\mathcal{M}^\bullet$ with compact support by

$$\mathrm{FT}^{\pm}_{\dagger}(\mathcal{M}^\bullet) := \mathrm{pr}_{s\dagger}(\mathrm{pr}_t^+ \mathcal{M}^\bullet \otimes \mathcal{E}^{\pm t\cdot s}).$$

The two kinds of Fourier transforms are interchanged by the duality functor

$$\mathbb{D} \circ \mathrm{FT}^{-1} = \mathrm{FT}_\dagger \circ \mathbb{D}, \tag{3.3.1}$$

and the forget support morphism induces an isomorphism $\mathrm{FT}_\dagger(\mathcal{M}) \xrightarrow{\sim} \mathrm{FT}(\mathcal{M})$, see [Mal91, Appendice 2 Proposition 1.7], so that we also have

$$\mathbb{D} \circ \mathrm{FT}^{-1} = \mathrm{FT} \circ \mathbb{D}. \tag{3.3.2}$$

Moreover $\mathrm{FT}^{-1} \circ \mathrm{FT} = \mathrm{id}$.

Proposition 3.7. For any $N \in \mathrm{Mod}_h(\mathscr{D}_Y)$ we have isomorphisms in $D^b(\mathscr{D}_{\mathbb{A}^1_s})$:

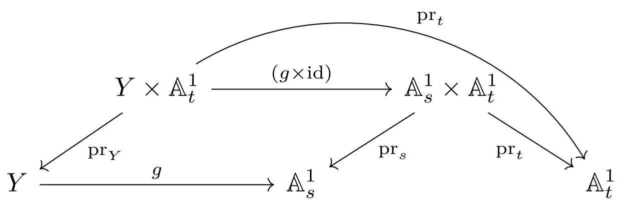
$$\mathrm{FT}^{-1}(\mathrm{pr}_{t+}(\mathrm{pr}_Y^+ N \otimes \mathcal{E}^{t\cdot g})) \cong g_+ N$$

and

$$\mathrm{FT}^{-1}(\mathrm{pr}_{t\dagger}(\mathrm{pr}_Y^+ N \otimes \mathcal{E}^{t\cdot g})) \cong g_\dagger N,$$

here we see g as a morphism $g\colon X \to \mathbb{A}^1_s$.

Proof. It suffices to calculate $\mathrm{FT}(g_+N)$ and $\mathrm{FT}(g_\dagger N)$. Consider the following diagram:



Then we have

$$\begin{aligned}
\mathrm{FT}\,(g_+N) &= \mathrm{pr}_{t+}\left(\mathrm{pr}_s^+ g_+ N \otimes \mathcal{E}^{s\cdot t}\right) \\
&= \mathrm{pr}_{t+}\left((g\times \mathrm{id}_t)_+ \mathrm{pr}_Y^+ N \otimes \mathcal{E}^{s\cdot t}\right) && \text{smooth base change}\\
&= \mathrm{pr}_{t+} \circ (g\times \mathrm{id}_t)_+ \left(\mathrm{pr}_Y^+ N \otimes (g\times \mathrm{id}_t)^+ \mathcal{E}^{s\cdot t}\right) && \text{projection formula}\\
&= \mathrm{pr}_{t+} \circ (g\times \mathrm{id}_t)_+ \left(\mathrm{pr}_Y^+ N \otimes \mathcal{E}^{g\cdot t}\right) && \text{definition of } \mathcal{E}^{s\cdot t}\\
&= \mathrm{pr}_{t+}\left(\mathrm{pr}_Y^+ N \otimes \mathcal{E}^{g\cdot t}\right).
\end{aligned}$$

In the same way, we obtain

$$\mathrm{FT}^{-1}\,(g_+N) = \mathrm{pr}_{t+}\left(\mathrm{pr}_Y^+ N \otimes \mathcal{E}^{-g\cdot t}\right). \tag{3.3.3}$$

Using the usual commutation rules of the duality functor with direct and inverse images, as well as the commutation of duality with Fourier transformation discussed above, we derive from the previous computation the following:

$$\begin{aligned}
\mathrm{FT}(g_\dagger \mathbb{D}_Y(N)) &= \mathrm{FT}(\mathbb{D}_{\mathbb{A}^1_s}(g_+N)) \\
&= \mathbb{D}_{\mathbb{A}^1_t}(\mathrm{FT}^{-1}(g_+N)) && \text{use formula (3.3.2)}\\
&= \mathbb{D}_{\mathbb{A}^1_t}\, \mathrm{pr}_{t+}\left(\mathrm{pr}_Y^+ N \otimes \mathcal{E}^{-g\cdot t}\right) && \text{use formula (3.3.3)}\\
&= \mathrm{pr}_{t\dagger}\, \mathbb{D}_{Y\times \mathbb{A}^1_t}\left(\mathrm{pr}_Y^+ N \otimes \mathcal{E}^{-g\cdot t}\right) \\
&= \mathrm{pr}_{t\dagger}\left(\mathrm{pr}_Y^+ \mathbb{D}_Y N \otimes \mathbb{D}_{Y\times \mathbb{A}^1_t}\mathcal{E}^{-g\cdot t}\right) \\
&= \mathrm{pr}_{t\dagger}\left(\mathrm{pr}_Y^+ \mathbb{D}_Y N \otimes \mathcal{E}^{g\cdot t}\right) && \mathcal{E}^{-g\cdot t} \text{ is locally free.}
\end{aligned}$$

Since this holds for any N, we obtain $\mathrm{FT}(g_\dagger N) = \mathrm{pr}_{t\dagger}\left(\mathrm{pr}_Y^+ N \otimes \mathcal{E}^{g\cdot t}\right)$, as required. □

Let $j\colon \mathbb{G}_{m,t} \hookrightarrow \mathbb{A}^1_t$ be the inclusion and let N be a $\mathscr{D}_Y$-module. Consider

$$\mathcal{F} := j^+ \operatorname{pr}_{t+}(\mathcal{E}^{t\cdot g} \otimes \operatorname{pr}_Y^+ N),$$

which is a priori an object in $D^b(\mathscr{D}_{\mathbb{G}_{m,t}})$. Recall that for $a \in \mathbb{Q}$ and $\chi = \exp(2\pi i a)$, we write $\mathbb{L}_\chi := (\mathcal{O}_{\mathbb{G}_{m,t}}, \mathrm{d} + a\frac{\mathrm{d}t}{t})$ and we put $\mathcal{F}_\chi := \mathcal{F} \otimes \mathbb{L}_\chi$. We will consider the following geometric situation:

Assumption 3.8. We take $a \in \mathbb{Q} \setminus \mathbb{Z}$, i.e. $\chi \neq 1$, and subject the above objects to the following four conditions.

1. $\mathcal{F}$ is a $\mathscr{D}_{\mathbb{G}_{m,t}}$-module concentrated in degree 0, regular singular at 0, and irregular singular at ∞ with slopes 1. Notice that the same assumption then automatically holds for $\mathcal{F}_\chi$.
2. The forget support morphism
$$j^+\operatorname{pr}_{t\dagger}(\mathcal{E}^{\pm t\cdot g} \otimes \operatorname{pr}_Y^+ N) \to j^+\operatorname{pr}_{t+}(\mathcal{E}^{\pm t\cdot g} \otimes \operatorname{pr}_Y^+ N) = \mathcal{F} \tag{3.3.4}$$
induces an isomorphism of $\mathscr{D}_{\mathbb{G}_{m,t}}$-modules concentrated in degree 0.
3. The morphism
$$j_\dagger(\mathcal{F}_\chi) \to j_+(\mathcal{F}_\chi) \tag{3.3.5}$$
is an isomorphism, so in particular both $j_\dagger(\mathcal{F}_\chi)$ and $j_+(\mathcal{F}_\chi)$ are isomorphic to $j_{\dagger+}(\mathcal{F}_\chi)$.
4. N underlies a pure Hodge module $N^{\mathrm{H}} \in \mathrm{MHM}(Y)$ of weight $\dim Y$.

$\diamondsuit$

We continue our discussion of various Fourier transforms. We restrict, however, from now on our attention to objects satisfying the above four conditions.

For two $\mathscr{D}_{\mathbb{A}^1}$-modules M and N, the additive convolutions are $\mathscr{D}_{\mathbb{A}^1}$-modules defined as

$$M \star_+ N := \operatorname{sum}_+(M \boxtimes N) \quad \text{and} \quad M \star_\dagger N := \mathbb{D}(\mathbb{D}(M) \star_+ \mathbb{D}(N)),$$

where $\operatorname{sum}\colon \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ is the summation map.

Proposition 3.9. Under Assumption 3.8 we have

$$\mathrm{FT}^{-1}(j_+\mathcal{F}_\chi) = g_+N \star_+ j_+\mathbb{L}_{\chi^{-1}} \tag{3.3.6}$$

and

$$\mathrm{FT}^{-1}(j_\dagger\mathcal{F}_\chi) = g_\dagger N \star_\dagger j_\dagger \mathbb{L}_{\chi^{-1}}. \tag{3.3.7}$$

Here $\star_+$ and $\star_\dagger$ are the additive convolutions.

Proof. Since the Fourier transform interchanges the tensor product and the additive convolution, by [DS13, Equation (1.1.2)] and the first equation in Proposition 3.7, we have

$$\begin{aligned}&\mathrm{FT}^{-1}(\operatorname{pr}_{t+}(\operatorname{pr}_Y^+ N \otimes \mathcal{E}^{t\cdot g}) \otimes j_+\mathbb{L}_\chi)\\ &\quad = \mathrm{FT}^{-1}(\operatorname{pr}_{t+}(\operatorname{pr}_Y^+ N \otimes \mathcal{E}^{t\cdot g})) \star_+ \mathrm{FT}^{-1}(j_+\mathbb{L}_\chi)\\ &\quad = g_+N \star_+ j_+\mathbb{L}_{\chi^{-1}}.\end{aligned}$$

Notice that the adjunction morphism $\mathrm{id} \to j_+j^+$ induces a morphism

$$\operatorname{pr}_{t+}(\operatorname{pr}_Y^+ N \otimes \mathcal{E}^{t\cdot g}) \to j_+\mathcal{F}$$

whose kernel and cokernel are supported at 0. Hence, we deduce an isomorphism

$$\operatorname{pr}_{t+}(\operatorname{pr}_Y^+ N \otimes \mathcal{E}^{t\cdot g}) \otimes j_+\mathbb{L}_\chi \simeq j_+\mathcal{F}_\chi.$$

Combining the above identities, we deduce (3.3.6).

Similarly, we have

$$\mathrm{FT}(\mathrm{pr}_{t+}(\mathrm{pr}_Y^+ N \otimes \mathcal{E}^{-t\cdot g}) \otimes j_+\mathbb{L}_{\chi^{-1}}) = \mathrm{FT}(\mathrm{pr}_{t+}(\mathrm{pr}_Y^+ N \otimes \mathcal{E}^{-t\cdot g})) \star_+ \mathrm{FT}(j_+\mathbb{L}_\chi^{-1}) = g_+N \star_+ j_+\mathbb{L}_\chi.$$

Notice that

$$\mathrm{pr}_{t+}(\mathrm{pr}_Y^+ N \otimes \mathcal{E}^{-t\cdot g})|_{\mathbb{G}_m} \simeq \mathrm{pr}_{t\dagger}(\mathrm{pr}_Y^+ N \otimes \mathcal{E}^{-t\cdot g})|_{\mathbb{G}_m} = \mathbb{D}_{\mathbb{G}_m}(\mathcal{F}) = \mathcal{F}^\vee,$$

we argue similarly above that

$$\mathrm{pr}_{t+}(\mathrm{pr}_Y^+ N \otimes \mathcal{E}^{-t\cdot g}) \otimes j_+\mathbb{L}_{\chi^{-1}} = j_+(\mathcal{F}_\chi)^\vee.$$

So, we deduce that

$$\mathrm{FT}(j_+(\mathcal{F}_\chi)^\vee) = g_+N \star_+ j_+\mathbb{L}_\chi$$

Applying the duality functor, we have

$$\begin{aligned}\mathrm{FT}_\dagger^{-1}(j_\dagger\mathcal{F}_\chi) &= \mathbb{D}_{\mathbb{A}^1} \circ \mathrm{FT}(j_+(\mathcal{F}_\chi)^\vee)\\ &= \mathbb{D}_{\mathbb{A}^1}(g_+N \star_+ j_+\mathbb{L}_\chi)\\ &= \mathbb{D}_{\mathbb{A}^1}(\mathbb{D}_{\mathbb{A}^1}(g_\dagger N) \star_+ \mathbb{D}_{\mathbb{A}^1}(j_\dagger\mathbb{L}_{\chi^{-1}}))\\ &= g_\dagger N \star_\dagger j_\dagger\mathbb{L}_{\chi^{-1}},\end{aligned}$$

which is (3.3.7). □

Mixed Hodge modules on the inverse Fourier transformation Notice that $g_\dagger N \star_\dagger j_\dagger\mathbb{L}_\chi$ resp. $g_+N\star_+ j_+\mathbb{L}_\chi$ underlies a mixed Hodge module

$$\mathrm{sum}_!\big(g_!N^{\mathrm{H}} \boxtimes j_!\mathbb{L}_\chi^{\mathrm{H}}\big) \quad \text{resp.} \quad \mathrm{sum}_*\big(g_*N^{\mathrm{H}} \boxtimes j_*\mathbb{L}_\chi^{\mathrm{H}}\big),$$

where N^{H} is the pure Hodge module on Y of weight $\dim Y$ with underlying $\mathscr{D}$-module N, and $\mathbb{L}_\chi^{\mathrm{H}}$ is the pure complex Hodge module on $\mathbb{G}_m$ of weight 1 with underlying $\mathscr{D}$-module $\mathbb{L}_\chi$. We therefore obtain the following.

Lemma 3.10. Under Assumption 3.8, the $\mathscr{D}_{\mathbb{A}^1}$-module $g_+N \star_+ j_+\mathbb{L}_\chi$ underlies a pure Hodge module of weight $\dim Y + 1$ on $\mathbb{A}^1$.

Proof. It follows from (3.3.4), (3.3.5), (3.3.6), and (3.3.7) that the forget support morphism

$$g_\dagger N \star_\dagger j_\dagger\mathbb{L}_\chi \to g_+N \star_+ j_+\mathbb{L}_\chi, \tag{3.3.8}$$

being the composition

$$\mathrm{FT}^{-1}(j_\dagger\mathcal{F}_\chi) \to \mathrm{FT}^{-1}(j_{\dagger+}\mathcal{F}_\chi) \to \mathrm{FT}^{-1}(j_+\mathcal{F}_\chi),$$

is an isomorphism.

By the properties of conservation of weights of mixed Hodge modules (see [Sai89, Prop. 1.7]), the two mixed Hodge modules $\mathrm{sum}_!\big(g_!N^{\mathrm{H}} \boxtimes j_!\mathbb{L}_\chi^{\mathrm{H}}\big)$ and $\mathrm{sum}_*\big(g_*N^{\mathrm{H}} \boxtimes j_*\mathbb{L}_\chi^{\mathrm{H}}\big)$ are mixed of weight $\leq \dim Y + 1$ and $\geq \dim Y + 1$ respectively. Since (3.3.8) can be lifted to an isomorphism of mixed Hodge modules, we conclude that they are pure Hodge modules of weight $\dim Y + 1$. □

The case of Frenkel–Gross connections Let $\check{G}$ be a simply connected group (i.e., G an almost simple group of adjoint type) and V a representation of $\check{G}$. We use notation from (2.4.1) and consider the case that $X = \mathrm{Gr}^\circ$, $g = F(x,1)$, pr_t is the projection to $\mathbb{A}^1_t$ such that $\mathrm{pr}_t \circ (\mathrm{id}\times j) = j \circ \pi^\circ$, $N = \mathrm{IC}_V|_{\mathrm{Gr}^\circ}$ as follows:

$$\begin{array}{ccccc} & \mathrm{Gr}^\circ \times \mathbb{G}_m & \xrightarrow{\mathrm{id}\times j} & \mathrm{Gr}^\circ \times \mathbb{A}^1_t & \\ {\scriptstyle t\cdot g}\swarrow & & \searrow{\scriptstyle \pi^\circ} & & \searrow{\scriptstyle \mathrm{pr}_t} \\ \mathbb{A}^1 & & & \mathbb{G}_m & \xrightarrow{j} \mathbb{A}^1_t. \end{array} \tag{3.3.9}$$

From the discussion above, we deduce the following corollary:

Corollary 3.11. The inverse Fourier transformation $\mathrm{FT}^{-1} \circ j_+(\widetilde{\nabla}_{\check{G}}(V) \otimes \mathbb{L}_\chi)$ is a regular holonomic $\mathscr{D}_{\mathbb{A}^1}$-module. If $\chi \neq 1$, it can be lifted to a pure Hodge module $\widetilde{M}^{\mathrm{H}}_\chi$ on $\mathbb{A}^1$ of weight $w := \dim \mathrm{Gr}_{\lambda^\vee} + 1 = \langle 2\rho, \lambda^\vee \rangle + 1$.

Proof. First note that $t \cdot g$ can be extended to $\mathrm{Gr}^\circ \times \mathbb{A}^1_t$. Recall that $\mathrm{IC}_{V,\mathrm{GR}^\circ} = \mathrm{pr}_Y^+ \mathrm{IC}_{V,\mathrm{Gr}^\circ}$ via the projection $\mathrm{pr}_Y : \mathrm{GR}^\circ = \mathrm{Gr}^\circ \times \mathbb{G}_m \to \mathrm{Gr}^\circ$. It then follows from Proposition 2.5 that

$$\widetilde{\nabla}_{\check{G}}(V) = \pi^\circ_+(\mathcal{E}^{t\cdot g} \otimes \mathrm{pr}_Y^+ \mathrm{IC}_{V,\mathrm{Gr}^\circ}).$$

It can be verified that the conditions of Assumption 3.8, as outlined at the beginning of Section 3.3, are satisfied for this choice of $(Y, g, N, \mathcal{F} = \widetilde{\nabla}_{\check{G}}(V) \otimes \mathbb{L}_\chi)$. Indeed, given our choices, the first and last conditions are straightforward. The second condition is verified by (2.3.2), and the third condition holds because ∇_G is regular singular and has principal unipotent monodromy at 0.

It then follows from Proposition 3.9 and Lemma 3.10 that $\mathrm{FT}^{-1} \circ j_+(\widetilde{\nabla}_{\check{G}}(V) \otimes \mathbb{L}_\chi)$ underlies a pure Hodge module on $\mathbb{A}^1$ of weight announced in the statement of the corollary. □

Corollary 3.12. Assume that $\nabla_{\check{G}}(V)$ is an irreducible connection on $\mathbb{G}_{m,t}$. Then there is a unique rescalable integrable mixed twistor $\mathscr{D}$-modules of weight 0 on $\mathbb{A}^1_t$ with underlying $\mathscr{D}_{\mathbb{A}^1_t}$-module equal to $j_{\dagger+}\tilde{\nabla}_{\check{G}}(V)$. Furthermore, $\mathbf{FT} \circ v\left(\widetilde{M}^{\mathrm{H}}_\chi\right)$ coincides with the rescalable mixed twistor $\mathscr{D}$-module up to a Tate twist defined in (3.2.4), where $\mathbf{FT}$ is the Fourier transform for $\mathrm{MTM}^{\mathrm{int}}(\mathbb{A}^1)$.

Proof. Apply Lemma 3.4 for $Y = \mathbb{A}^1$ and for $N = j_{\dagger+}\tilde{\nabla}_{\check{G}}(V)$ as well as for $N = j_+(\tilde{\nabla}_{\check{G}}(V) \otimes \mathbb{L}_\chi)$. □

3.4 The irregular Hodge numbers of $\nabla_{\check{G}}$

This section aims to prove Theorem 1.1. We first make some reductions on the proof of Theorem 1.1 in the following lemmas:

Lemma 3.13. Let G_1 and G_2 be two isogenous almost simple groups. If the irregular Hodge filtrations of $\nabla^{\mathrm{H}}_{\check{G}_1}$ are determined by ρ_{G_1}, then the irregular Hodge filtrations of $\nabla^{\mathrm{H}}_{\check{G}_2}$ are determined by ρ_{G_2}.

Proof. Without loss of generality, we may assume that we have an isogeny $\pi : \check{G}_1 \to \check{G}_2$ or $\pi : \check{G}_2 \to \check{G}_1$. In the former case, we deduce the irregular Hodge filtrations of $\nabla^{\mathrm{H}}_{\check{G}_2}$ by the 'trivial' functoriality, see the discussion in Section 2.2.

In the latter case, assume that the irregular Hodge filtration of $\nabla^{\mathrm{H}}_{\check{G}_2}$ is determined by η/k. By the 'trivial' functoriality again, η/k will induce $2\rho_{G_1}$ on $\check{G}_1$, i.e.,

$$\pi \circ \eta^2 = (2\rho_{G_1})^k.$$

Hence, the cocharacter

$$\pi \circ (\eta^2 \cdot (2\rho_{G_2})^{-k})$$

is trivial, which forces $\eta^2 \cdot (2\rho_{G_2})^{-k}$ to take values in $\ker \pi$. Since there is no nontrivial homomorphism from $\mathbb{G}_m$ to finite groups, we conclude that

$$\eta^2 = (2\rho_{G_2})^k,$$

which means that $\eta/k = \rho_{G_2}$. □

Lemma 3.14. If the irregular Hodge numbers of $\nabla^{\mathrm{H}}_{\check{G}}(V)$ are given by $\rho(V)$ for one faithful representation V of $\check{G}$, then the irregular Hodge numbers of $\nabla^{\mathrm{H}}_{\check{G}}$ are given by ρ.

Proof. As algebraic vector bundles on $\mathbb{G}_m$ are trivial, we can (non-canonically) identify any basis of $\nabla_{\check{G}}(L)$ with a basis of L for any representation L. In particular, the irregular Hodge filtration on $\nabla_{\check{G}}(L)$ determines a flag (indexed by $\frac{1}{k}\mathbb{Z}$)

$$\mathscr{F}_{\alpha/k} \subset \ldots \nabla_{\check{G}}(L)$$

for $\alpha \in \mathbb{Z}$, such that $h^\alpha = \mathrm{rk}\, \mathscr{F}_{\alpha/k}/\mathscr{F}_{(\alpha-1)/k}$.

By Proposition 3.6, the irregular Hodge filtration is determined by η/k for some cocharacter η of $\check{G}$ and an integer k. It follows that $\eta(V)$ and $\rho^k(V)$ determines the same flag of V. As V is faithful, η and ρ^k correspond to the same parabolic subgroup $\check{P}$ of $\check{G}$ up to conjugacy. It follows that for any representation W of $\check{G}$, $\eta(W)$ and $\rho^k(W)$ determine again the same flag on W. Hence, the irregular Hodge numbers of $\nabla_{\check{G}}(W)$ are determined by $\rho(W)$. □

Now, we need to calculate the irregular Hodge numbers of some specific representations V. Similar to [Qin24, §5.2.1], we consider the case where the local monodromy of $\nabla_{\check{G}}(V)$ at 0 consists of Jordan blocks of different sizes.

Proposition 3.15. Assume that V is irreducible and the local monodromy of $\nabla_{\check{G}}(V)$ at 0 consists of Jordan blocks of sizes $r_1 < r_2 < \cdots < r_k$, then up to a global shift, the irregular Hodge numbers of $\nabla_{\check{G}}(V)$ are given by

$$h^p_{\mathrm{irr}} := \#\{(i, a) \mid 2p = r_i + w - 1 - 2a,\ 0 \le a \le r_i\}$$

for $p \in \mathbb{Z}$, where $w := \langle 2\rho, \lambda^\vee\rangle + 1$.

Proof. Let $\widetilde{\nabla}_{\check{G}}(V) = [h]^+\nabla_{\check{G}}(V)$, $\mathbb{L}_\chi$ and g be as in the previous section. In order to prove the proposition, it suffices to calculate the irregular Hodge numbers of $\widetilde{\nabla}_{\check{G}}(V) \otimes \mathbb{L}_\chi$ for any $\chi \neq 1$.

Recall that $j\colon \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ denotes the canonical embedding. Notice that $\widetilde{\nabla}_{\check{G}}(V) \otimes \mathbb{L}_\chi$ is regular at 0, with quasiunipotent monodromy of eigenvalue χ. So we have

$$j_\dagger(\widetilde{\nabla}_{\check{G}}(V) \otimes \mathbb{L}_\chi) = j_{\dagger+}(\widetilde{\nabla}_{\check{G}}(V) \otimes \mathbb{L}_\chi) = j_+(\widetilde{\nabla}_{\check{G}}(V) \otimes \mathbb{L}_\chi).$$

Recall also that $\widetilde{M}_\chi := \mathrm{FT}^{-1} j_{\dagger+}(\widetilde{\nabla}_{\check{G}}(V) \otimes \mathbb{L}_\chi)$ is the inverse Fourier transform of $j_{\dagger+}(\widetilde{\nabla}_{\check{G}}(V) \otimes \mathbb{L}_\chi)$, which can be lifted to a pure Hodge module $\widetilde{M}^{\mathrm{H}}_\chi$ by Corollary 3.11.

Let $\psi_t\,(\widetilde{\nabla}_{\check{G}}(V) \otimes \mathbb{L}_\chi)$ be the nearby cycle module at 0 of $j_+(\widetilde{\nabla}_{\check{G}}(V) \otimes \mathbb{L}_\chi)$. Notice that the local monodromy of $\widetilde{\nabla}_{\check{G}}(V)$ at 0 is also unipotent with Jordan blocks of sizes $r_1 < \cdots < r_k$. So that of $\psi_t\,(\widetilde{\nabla}_{\check{G}}(V) \otimes \mathbb{L}_\chi)$ is quasiunipotent with eigenvalue χ and with Jordan blocks of sizes $r_1 < \cdots < r_k$.

By construction, we have $\widetilde{M}_\chi = \mathrm{FT}^{-1} j_{\dagger+}(\widetilde{\nabla}_{\check{G}}(V) \otimes \mathbb{L}_\chi)$. Applying the (inverse) stationary phase formula (cf. [Mal91, Prop. VII.2.4] or [Sab06, Prop. 2.3]), it follows that

$$\psi_t\,(\widetilde{\nabla}_{\check{G}}(V) \otimes \mathbb{L}_\chi) = \phi_{1/\tau}\widetilde{M}_\chi = \psi_{1/\tau,-1}\widetilde{M}_\chi \tag{3.4.1}$$

and the corresponding nilpotent part of the monodromy operator N has Jordan blocks of sizes $r_1 < \cdots < r_k$ respectively. In particular, $\mathrm{rk}\ \widetilde{M}_\chi = r$.

By our assumption, the (non-zero) primitive parts of the Lefschetz decomposition of $\psi_{1/\tau}\widetilde{M}^{\mathrm{H}}_\chi$, denoted by $P_{r_1}, \ldots, P_{r_k}$, are one-dimensional and are of Hodge–Tate type. Since $\widetilde{M}^{\mathrm{H}}_\chi$ is pure of weight $w = \langle 2\rho, \lambda^\vee\rangle + 1$, the primitive parts $P_{r_1}, \ldots, P_{r_k}$ are pure of weights $r_1 + w - 1, \ldots, r_k + w - 1$ respectively. In other words,

$$P_{r_i} = \mathrm{gr}^W_{r_i + w - 1} P_{r_i}$$

for $1 \le i \le k$. Then, by the Lefschetz decomposition, for each $\ell \in \mathbb{Z}$, the graded quotient $\mathrm{gr}^W_\ell \psi_{1/\tau}\widetilde{M}^{\mathrm{H}}_\chi$ is Hodge–Tate of dimension

$$\#\{(i, a) \mid \ell = r_i + w - 1 - 2a, 0 \le a \le r_i\}.$$

Moreover, we have

$$\dim \mathrm{gr}^p_F \psi_{1/\tau}\widetilde{M}^{\mathrm{H}}_\chi = \dim \mathrm{gr}^W_{2p} \psi_{1/\tau}\widetilde{M}^{\mathrm{H}}_\chi.$$

Since the eigenvalues of $\widetilde{M}^{\mathrm{H}}_\chi$ at ∞ are different from 1, we conclude using [SY19, (7)] that

$$\dim \mathrm{gr}^{p+a}_{F_{irr}}(\widetilde{\nabla}_{\check{G}}(V) \otimes \mathbb{L}_\chi) = \dim \mathrm{gr}^p_F \psi_{1/\tau,-1}\widetilde{M}^{\mathrm{H}}_\chi = \#\{(i, a) \mid 2p = r_i + w - 1 - 2a, 0 \le a \le r_i\},$$

where a is a rational number such that $\chi = \exp(2\pi i a)$. Notice that the irregular Hodge numbers of $\widetilde{\nabla}_{\check{G}}(V) \otimes \mathbb{L}_\chi$ agree with those of $\widetilde{\nabla}_{\check{G}}(V)$ (as well as $\nabla_{\check{G}}(V)$) up to a global shift. Therefore, we have shown the formula for the irregular Hodge numbers of $\nabla_{\check{G}}(V)$ as announced in the statement. □

Corollary 3.16. Assume that $\nabla_{\check{G}}(V)$ is irreducible and its local monodromy at 0 consists of Jordan blocks of different sizes. Then the Hodge numbers of $\nabla_{\check{G}}(V)$ are determined by $\rho(V)$.

Proof. Let $V = \bigoplus_{d\in\frac{1}{2}\mathbb{Z}} V_d$ be the degree decomposition of V with respect to $2\rho\colon \mathbb{G}_m \to \check{G}$, such that $2\rho(t)$ acts on V_d by multiplication by t^{2d}. There is a decreasing filtration induced by ρ, i.e., the filtration defined by

$$F^p_\rho := \bigoplus_{p\leq 2d} V_d$$

for $p \in \mathbb{Z}$. Equivalently, 2ρ can be seen as an element in $\check{\mathfrak{g}}$, which induces an endomorphism $2\rho(V)$ on V such that $v \in V_d$ if and only if $2\rho(V)\cdot v = 2dv$.

Notice that one can upgrade the principal nilpotent operator N to a $\mathfrak{sl}_2$-triple (E, F, H) such that $F = N$. Recall that ρ can equivalently be written as $\rho = \sum_{i=1}^n \omega_i$, the sum of *fundamental coweights* of $\check{G}$ satisfying $\langle\alpha_i^\vee, \omega_j\rangle = \delta_{ij}$ for $\{\alpha_1^\vee, \dots, \alpha_n^\vee\} = \Delta$ an ordering of the simple roots of $\check{G}$. Hence, we have

$$[N,\rho] = \textstyle\sum_{i,j=1}^n [X_{-\alpha_i^\vee}, \omega_j] = \sum_{i,j=1}^n \langle -\alpha_i^\vee, \omega_j\rangle\, X_{-\alpha_i^\vee} = -N. \tag{3.4.2}$$

So, we deduce that

$$H = 2\rho. \tag{3.4.3}$$

In particular, we have

$$\rho(V)N(V)v = N(V)\rho(V)v + N(V)v = (d+1)N(V)v$$

for $v \in V_d$. Hence, one has for $d \in \frac{1}{2}\mathbb{Z}$ that

$$N(V)F^{2d}_\rho V \subset F^{2d+2}_\rho V.$$

On the other hand, viewing V as the representation of $\mathfrak{sl}_2$ via the composition $\mathfrak{sl}_2 \to \check{\mathfrak{g}} \to \mathfrak{gl}(V)$, we have

$$N(V)^{2d}\colon \mathrm{gr}^{-2d}_{F_\rho} V = V_{-d} \xrightarrow{\sim} V_d = \mathrm{gr}^{2d}_{F_\rho} V$$

when $d \leq 0$ by the representation theory of $\mathfrak{sl}_2$ [Hum72, p.33 Thm.]. So $F^{2\bullet}_\rho$ is the same as the monodromy-weight filtration on V with respect to $N(V)$.

Notice that $\psi_t\,(\widetilde{\nabla}_{\check{G}}(V) \otimes \mathbb{L}_\chi)$ with its monodromy-weight filtration is abstractly isomorphic to V with its monodromy-weight filtration with respect to $N(V)$, and so is $\psi_{1/\tau,-1}\widetilde{M}^{\mathrm{H}}_\chi$ by the identification from (3.4.1). Therefore, combining with Proposition 3.15, the irregular Hodge numbers of $\nabla_{\check{G}}(V)$ are given by $\rho(V)$. □

To achieve the proof of Theorem 1.1, it suffices to show the existence of representations V such that the Jordan blocks of $N(V)$ have different sizes. In fact, adjoint representations are what we are looking for, as suggested by the following lemma:

Lemma 3.17. For a simple group $\check{G}$, the nilpotent operator N acting on the adjoint representation $V = \mathfrak{g}$ has Jordan blocks of different sizes.

Proof. Let V be the adjoint representation $V = \check{\mathfrak{g}} = \mathrm{Lie}(\check{G})$ of $\check{G}$ and l be the rank of $\check{\mathfrak{g}}$. Notice that $N \in \check{\mathfrak{g}}$ is principally nilpotent, which can be enhanced into a principal $\mathfrak{sl}_2$. By [Kon59, §6.5 & Cor. 8.7], we can decompose $V = \check{\mathfrak{g}}$ into $\bigoplus_i V_{k_i}$, where $\dim V_{k_i}$ are $(2k_i+1)$-dimensional irreducible $\mathfrak{sl}_2$-representations, and the numbers $n_1 \leq \ldots \leq n_l$ are the *exponents* of $\check{\mathfrak{g}}$.

By [Bou02, Plate I-VII], the exponents of $\check{\mathfrak{g}}$ are distinct numbers. So the dimensions of irreducible representations (as well as the Jordan blocks of N) are distinct numbers. □

Proof of Theorem 1.1 The rescalable integrable mixed twistor $\mathscr{D}$-module on $\nabla_{\check{G}}$ exists by Proposition 3.6. To calculate its irregular Hodge numbers, it suffices to prove the theorem for one type in each row of diagram (2.2.1) by the functoriality (2.2.2). So we assume that $\check{G}$ is of type A_n, B_n, E_7, E_8, and F_4.

For each type, by Lemma 3.13, it suffices to prove the theorem for one almost simple group. So we prove the theorem for simple groups for each type.

By Lemma 3.14, it suffices to know the Hodge numbers of $\nabla_{\check{G}}(V)$ for one faithful representation of $\check{G}$. So we choose V as the adjoint representation of these groups. From Lemma 3.17, as $\nabla_{\check{G}}(V)$ is irreducible and its local monodromy at 0 have Jordan blocks of different sizes, we conclude the proof by Corollary 3.16.

4 On a conjecture of Katzarkov–Kontsevich–Pantev

By a *Landau–Ginzburg model*, we mean a pair (Y, w) consisting of a quasi-projective variety Y and a regular function $w\colon Y \to \mathbb{A}^1$. In [KKP17, Conjecture 3.7.], the authors defined *Landau–Ginzburg Hodge numbers* of (Y, w) as

$$f^{p,q}(Y, \mathsf{w}) := \dim_{\mathbb{C}} \mathrm{H}^p(\mathsf{Z}, \Omega^q_{\mathsf{Z}}(\log D, w)),$$

where Z is a smooth proper compactification of Y such that $D := Z\backslash Y$ is a simple normal crossing divisor and w extends to a morphism $\tilde{w}\colon Z \to \mathbb{P}^1$, and $\Omega^q_Z(\log D, w)$ is the sheaf

$$\ker(\Omega^q_Z(\log D) \xrightarrow{dw\wedge} \Omega^{q+1}_Z(\log D)).$$

It is known ([ESY17, Theorem 1.3.2]) that we have

$$f^{p,q}(Y, \mathsf{w}) = \dim \mathrm{gr}^p_{F_{Yu}} \mathrm{H}^{p+q}_{\mathrm{dR}}(\Omega^\bullet_Y, d + dw\wedge),$$

where $F^\bullet_{Yu}$ is the Yu filtration on the twisted de Rham cohomologies as defined in [Yu14]. In particular, as the Yu filtration is independent of the choice of a compactification Z, so are the numbers $f^{p,q}(Y, w)$.

The authors of [KKP17] predicted that when a Landau–Ginzburg model (Y, w) is the mirror of a Fano variety X, the Hodge numbers of X are related to the Landau–Ginzburg Hodge numbers of $(Y, \mathbf{w})$ by the formula

$$f^{p,q}(Y, \mathsf{w}) = h^{p,n-q}(X), \tag{4.0.1}$$

where $n = \dim X$.

This conjecture has been verified in a number of cases, including the case of convenient and non-degenerate Laurent polynomials ([Sab18b, Thm. 3.6], where the variant [KKP17, Conjecture 3.6] is shown).

We prove this conjecture when $X = \check{G}/\check{P}$ is a minuscule homogeneous space. The mirror Landau–Ginzburg model is $(Y = \overset{\circ}{G/P}, w)$, where G resp. P are Langlands dual to $\check{G}$ resp. $\check{P}$ (subject to the choice of a root datum), $\overset{\circ}{G/P} \subset G/P$ is the open projected Richardson variety, and w is induced by some decoration function, see [LT24, §1.4].

Theorem 4.1. *For $X = \check{G}/\check{P}$ a minuscule homogeneous space with $\dim(X) = n$, consider the Landau-Ginzburg model (Y, w) from above. Then (4.0.1) holds, namely:*

$$f^{p,q}(Y, \mathsf{w}) = h^{p,n-q}(X) = \begin{cases} 0, & \text{if } q \neq n-p, \\ h^{p-\frac{n}{2}}_{irr}, & \text{if } q = n-p. \end{cases}$$

In particular, both numbers are determined by $\rho = 2\rho/2$ up to a shift.

Before giving the proof, we will explain how the Hodge numbers $f^{p,q}(Y, w)$ are related to our irregular Hodge numbers h_{irr} of Frenkel–Gross connections.

Lemma 4.2. The numbers $f^{p,q}(Y, w)$ are zero if $q \neq n - p$ and are equal to $h^{p-n/2}_{\mathrm{irr}}$ when $q = n - p$, which is determined by ρ up to a twist.

Proof. Following [LT24, §1.3], let $X_{(G,P)}$ be the *(parabolic) geometric crystal*, $f\colon X \to \mathbb{A}^1$ the *decoration function*, and $\pi\colon X_{(G,P)} \to \mathbb{G}_m$ the *highest weight function*. The fiber $X_t = \pi^{-1}(t)$ for $t \in \mathbb{G}_m(\mathbb{C})$ is called the *geometric crystal with highest weight* t, and is identified with the open projected Richardson variety $\overset{\circ}{G/P} \subset G/P$. Moreover, $(\overset{\circ}{G/P}, f|_{X_1})$ is the mirror Landau–Ginzburg model of $\check{G}/\check{P}$ as mentioned above.

The *character $\mathscr{D}$-module* of the geometric crystal $X_{(G,P)}$ is defined as

$$\mathrm{Cr}_{G,P} := \pi_+ \mathcal{E}^f. \tag{4.0.2}$$

According to [LT24, Thm. 1.8] and Theorem 2.4, the character $\mathscr{D}$-module $\mathrm{Cr}_{G,P}$ is a $\mathscr{D}$-module concentrated in degree 0 and is smooth on $\mathbb{G}_m$ such that

$$\mathrm{Cr}_{G,P} \simeq \nabla_{\check{G}}(V_{\lambda_P}), \tag{4.0.3}$$

where λ_P is the minuscule weight of the dual group $\check{G}$ of G determined by the parabolic subgroup P, and V_{λ_P} is the representation of the highest weight λ_P. In particular, the fiber of $\nabla_{\check{G}}(V_{\lambda_P})$ at 1 is the twisted de Rham cohomology $\mathbb{H}^n(\Omega_Y^\bullet, \mathrm{d}+\mathrm{d}w)$ associated with the Landau–Ginzburg model (Y, w).

By Theorem 3.3, we can view the rescalable integrable mixed twistor $\mathscr{D}$-module $\nabla^{\mathrm{H}}_{\check{G}}(V)$ in (3.2.1) as an exponential mixed Hodge module. By (4.0.3), its de Rham fiber is identified with $\mathrm{Cr}_{G,P}$. The fiber (or the pullback) of $\nabla^{\mathrm{H}}_{\check{G}}(V)$ at the smooth point 1 is an exponential mixed Hodge module (or structure) $\nabla^{\mathrm{H}}_{\check{G}}(V)_1$ of weight 0, whose de Rham fiber is $(\mathrm{Cr}_{G,P})_1 \simeq \mathbb{H}^n(\Omega_Y^\bullet, \mathrm{d}+\mathrm{d}w)(n/2)$, see also [FSY22, Def. A.18].

By [Moc21, Prop 11.22], the irregular Hodge filtration F_{irr} of $\nabla^{\mathrm{H}}_{\check{G}}(V)$ induces that of $\nabla^{\mathrm{H}}_{\check{G}}(V)_1$. Moreover, the irregular Hodge filtration on the de Rham fiber of $\nabla^{\mathrm{H}}_{\check{G}}(V)_1$ coincides with the Yu filtration on the twisted de Rham cohomology [ESY17, Prop 1.7.4] with a shift by $n/2$. Therefore, we have

$$h_{\mathrm{irr}}^{p-\frac{n}{2}} = \dim \mathrm{gr}^p_{F_{\mathrm{Yu}}} \mathbb{H}^n(\Omega_Y^\bullet, \mathrm{d}+\mathrm{d}w) = f^{p,n-p}.$$

At last, since $\mathbb{H}^{p+q}(\Omega_Y^\bullet, \mathrm{d}+\mathrm{d}w) = 0$ when $p+q \neq n$, we have $f^{p,q} = 0$ when $q \neq n-p$. □

Proof of Theorem 4.1. It is well known ([Che94] or [CG10, Chap. 3]) that the cohomology of a homogeneous space $\check{G}/\check{P}$ is of Hodge–Tate type, i.e. $h^{p,n-q}(X) = 0$ if $q \neq n-p$ and $h^{p,p}(X) = b_{2p}(X)$. Hence, we need to identify the Betti numbers of X with our $h_{irr}^{p-\frac{n}{2}}$. Since the latter are given by ρ, it is sufficient to show that $\mathrm{H}^{2p}(X)$ can be identified with the eigenspace of 2ρ acting on V, which under our assumptions is a minuscule representation of $\check{G}$. This is exactly what is stated in [LT24, Prop 4.12] (especially the last sentence of its proof). At last, by Lemma 4.2, we also have the identity $h^{p,n-q}(X) = f^{p,q}(Y, w)$. □

5 Examples of some small representations

In this subsection, we give examples of representations V such that $N(V)$ have Jordan blocks of different sizes. Following our strategy as explained in Section 3.4, this yields concrete results for the irregular Hodge numbers of Frenkel-Gross connections in these cases.

5.1 A_n

We take $V = \mathbb{C}^{n+1}$ as the standard representation of $\check{G} = \mathrm{SL}_{n+1}$. In this case the connection $\nabla_{\check{G}}(V)$ is the connection corresponding to the Bessel differential equation given in Example 2.1, and we observe that $N(V)$ is conjugate to a Jordan block of size $n+1$ with eigenvalues 0. So the local monodromy of $\nabla_{\check{G}}(V)$ at 0 consists of 1 single Jordan block, and the Hodge numbers are

$$h^\alpha = \begin{cases} 1, & \alpha \in \{-\frac{n+1}{2} + i \mid 0 \leq i \leq n+1\}, \\ 0, & \text{else}, \end{cases}$$

for $\alpha \in \frac{1}{2}\mathbb{Z}$. Our result does agree with the irregular Hodge numbers of the Kloosterman connection; see for example, [CnDRS19, SY19, QX23].

For the standard representation, the corresponding flag variety is the Projective space $\mathbb{P}^n$. By Theorem 4.1, we deduce that well-known formula of the Hodge numbers of $\mathbb{P}^n$

$$\{h^{p,p} \mid 0 \leq p \leq n\} = \{1, \ldots, 1\}.$$

5.2 B_n

Let $V = \mathbb{C}^{2n+1}$ be the standard representation of SO_{2n+1}. By [FG09, §6.3 Equation (5)], $N(V)$ is conjugated to a matrix with one Jordan block of size $2n+1$. So the local monodromy of $\nabla_{\check{G}}(V)$ at 0 consists of a single Jordan block of sizes $2n+1$ and the Hodge numbers are

$$h^\alpha = \begin{cases} 1, & |\alpha| \leq n-1, \\ 0, & \text{else}, \end{cases}$$

for $\alpha \in \mathbb{Z}$.

This also agrees with known examples. From our result for $\nabla_{\check{G}}(V)$, we deduce the irregular Hodge numbers of $\nabla_{\mathrm{SO}_{2n+1}}(V)$, which coincide with formulas given in [CnDS21, SY19, QX23] up to a shift.

5.3 E_6, F_4

Let V be a minuscule representation of E_6, and $\widetilde{V}$ the one of F_4. Recall that we have shown that $\nabla_{E_6}(V) = \nabla_{F_4}(\widetilde{V}) \oplus \mathcal{O}$ in (2.2.5).

To analyze the Jordan blocks of $N(V)$, we turn to the graph [SW23, (2.25)]. Each vertex represents a basis vector $v_i, v_i', $ or v_i'' of V. An edge with number k between a vertex $u := v_i, v_i'$, or v_i'' with anther vertex $w := v_{i+1}, v_{i+1}'$, or v_{i+1}'' means that $X_{\alpha_k} u$ is a non-zero multiple of w. Using the representation theory of $\mathfrak{sl}_2$, we deduce that the Jordan blocks of $N(V)$ are of sizes 19, 7, and 1 respectively. So the irregular Hodge numbers of $\nabla_{F_4}(\widetilde{V})$ and $\nabla_{E_6}(V)$ are

$$h^\alpha = \begin{cases} 1, & 4 < |\alpha| \leq 8, \\ 2, & |\alpha| \leq 4, \\ 0, & \text{else} \end{cases} \quad \text{and} \quad h^\alpha = \begin{cases} 1, & 4 < |\alpha| \leq 8, \\ 2, & 0 < |\alpha| \leq 4, \\ 3, & \alpha = 0, \\ 0, & \text{else}, \end{cases}$$

respectively, for $\alpha \in \mathbb{Z}$.

For the group E_6, the flag variety corresponds to the minuscule representation V is the Cayley plane. By Theorem 4.1, we deduce Hodge numbers of the Cayley plane are

$$\{h^{p,p} \mid 0 \leq p \leq 16\} = \{1,1,1,1,2,2,2,2,3,2,2,2,2,1,1,1,1\}.$$

5.4 E_7

Let V be a minuscule representation of E_7. By the graph [SW23, (2.27)], we deduce similarly as above from the representation theory of $\mathfrak{sl}_2$ that the Jordan blocks of $N(V)$ are of sizes 28, 18, and 10, respectively. So the irregular Hodge numbers of $\nabla_{E_7}(V)$ are

$$h^p = \begin{cases} 1, & 9 < |p| \leq 14, \\ 2, & 5 < |p| \leq 9, \\ 3, & |p| \leq 5, \\ 0, & \text{else}, \end{cases}$$

for $p \in \frac{1}{2} + \mathbb{Z}$.

For the group E_7, the flag variety corresponds to the minuscule representation V is the Freudenthal variety. By Theorem 4.1, we deduce that the Freudenthal variety has Hodge numbers

$$\{h^{p,p} \mid 0 \leq p \leq 27\} = \{1,1,1,1,1,2,2,2,2,3,3,3,3,3,3,3,3,3,3,2,2,2,2,1,1,1,1,1\}.$$

5.5 E_8

Let V be the adjoint representation $V = \mathfrak{e}_8$ of E_8 (of dimension 248). Notice that $N \in \mathfrak{e}_8$ is principally nilpotent, which can be enhanced into a principal $\mathfrak{sl}_2$. By [Kon59, §6.5 & Cor. 8.7], we can decompose $V = \mathfrak{e}_8$ into $\bigoplus_i V_{k_i}$, where $\dim V_{k_i}$ are $(2k_i + 1)$-dimensional irreducible $\mathfrak{sl}_2$-representations, and the numbers $n_1 < \cdots < n_8$ are the *exponents* of $\mathfrak{e}_8$.

By [Bou02, Place VII], the exponents of $\mathfrak{e}_8$ are $N_1 = 1, N_2 = 7, N_3 = 11, N_4 = 13, N_5 = 17, N_6 = 19, N_7 = 23$, and $N_8 = 29$ respectively. So the dimensions of irreducible representations (as well as the Jordan blocks of N) are of sizes $2N_i + 1$ for $i = 1, \ldots, 8$ respectively. So the irregular Hodge numbers of $\nabla_{E_8}(V)$ are

$$h^p = \begin{cases} 8, & |p| \leq 1 \\ 9 - i, & N_{i-1} < |p| \leq N_i \text{ for } 2 \leq i \leq 8 \\ 0, & \text{else} \end{cases}$$

for $p \in \mathbb{Z}$.

6 The $\mathcal{R}$-module associated to the Frenkel-Gross connection

We have established in the previous sections using some geometric arguments that the Frenkel-Gross-connection $\nabla_{\check{G}}$ can be upgraded to a tensor functor $\nabla^{\mathrm{H}}_{\check{G}}\colon \mathrm{Rep}(\check{G}) \to \mathrm{MTM}^{\mathrm{int}}_{\mathrm{resc}}(\mathbb{G}_{m,t})$ (see the discussion around Eq. (3.2.5)). Hence, for any $V \in \mathrm{Rep}(\check{G})$, the object $\nabla^{\mathrm{H}}_{\check{G}}(V)$ can in particular be considered as an integrable mixed twistor module on $\mathbb{G}_{m,t}$. Therefore, letting $\mathcal{R} := \mathcal{R}^{int}_{\mathbb{G}_{m,t}}$ be the sheaf of rings with global sections equal to $\mathbb{C}[z,t^{\pm}]\langle z^2\partial_z, z\partial_t\rangle$, we have a $\mathcal{R}$-triple $(\mathcal{M},\mathcal{M}',C)$, where $\mathcal{M},\mathcal{M}'$ are coherent $\mathcal{R}$-modules. Our conjecture concerns an explicit expression for $\mathcal{M}$. It seems difficult to establish this conjecture directly, however, if one could do so, the main result of this paper would follow almost immediately.

For an almost simple group $\check{G}$, we associate a tensor functor

$$\nabla^{\mathcal{R}}_{\check{G}}\colon \mathrm{Rep}(\check{G}) \to \mathrm{Mod}(\mathcal{R})$$

as follows. For each finite-dimensional complex representation V of $\check{G}$, let $\mathcal{E}(V) := \mathcal{O}_{\mathbb{A}^1_z\times\mathbb{G}_{m,t}} \otimes_{\mathbb{C}} V$ be the trivial bundle on $\mathbb{A}^1_z \times \mathbb{G}_{m,t}$, the associated connection on $\mathcal{E}(V)$ is given by

$$\nabla^{\mathcal{R}}_{\check{G}}(V) = \mathrm{d} + \big(N(V)+tE(V)\big)\frac{\mathrm{d}t}{tz} - \big(N(V)+tE(V)\big)h(G)\frac{\mathrm{d}z}{z^2} + \rho(V)\frac{\mathrm{d}z}{z}, \tag{6.0.1}$$

where $h(\check{G})$ is the Coxeter number of $\check{G}$, $N(V)$ and $E(V)$ are the endomorphisms of V induced by the action of $N, E \in \check{\mathfrak{g}}$ and $\rho(V)$ is the semisimple (i.e. diagonalizable) matrix induced by the action of $\rho = \frac{1}{2}\sum_{\alpha^\vee\in(\Phi^\vee)^+}\alpha^\vee$, half sum of positive coroots of $\check{G}$.

Notice that we have

$$\nabla^{\mathcal{R}}_{\check{G}}(V) : \mathcal{E}(V) \to \mathcal{E}(V) \otimes \frac{1}{z}\Omega^1_{\mathbb{A}^1_z\times\mathbb{G}_{m,t}}\left(\log(\{0\}\times\mathbb{G}_{m,t})\right).$$

Proposition 6.1. The pair $(\mathcal{E}(V), \nabla^{\mathcal{R}}_{\check{G}}(V))$ is integrable, i.e. we have $(\nabla^{\mathcal{R}}_{\check{G}}(V))^2 = 0$. In particular, the localization $(\mathcal{E}(V)(*(\{0\}\times\mathbb{G}_{m,t})), \nabla^{\mathcal{R}}_{\check{G}}(V))$ is a flat meromorphic connection, and yields a local system on $\mathbb{G}_{m,z}\times\mathbb{G}_{m,t}$.

Proof. First, recall that a connection ∇ of the form $\nabla = A\mathrm{d}t + B\mathrm{d}z$ (for $A, B \in \mathrm{Mat}(\dim(V)\times\dim(V), \mathcal{O}_{\mathbb{G}_{m,z}\times\mathbb{G}_{m,t}})$) is integrable if and only if A and B satisfy $\partial_z A - \partial_t B = [A,B]$. The connection in (6.0.1) is of this form with $A = \frac{1}{tz}(N+tE)$ and $B = -\frac{h}{z^2}(N+tE) + \frac{1}{z}\rho$. The left side of the condition is easy enough to evaluate: $\partial_z A - \partial_t B = \frac{1}{tz^2}\big(-N + t(h-1)E\big)$.

For the right side, we work inside the Lie algebra: $[A,B] = \frac{1}{tz^2}\big[N+tE,\rho\big]$. Recall that we have shown in (3.4.2) that $[N,\rho] = -N$. On the other hand, it is well-known that $\langle\theta,\rho\rangle = h-1$, so we find that $[E,\rho] = (h-1)E$.

Hence, we conclude that $[A,B] = \frac{1}{tz^2}\big(-N+t(h-1)E\big) = \partial_z A - \partial_t B$, i.e. that ∇ is integrable. □

We finish this paper by the following conjecture about an explicit expression of the irregular Hodge module structure defined on the Frenkel-Gross connection.

Conjecture 6.2. For any $V \in \mathrm{Rep}(\check{G})$, if $(\mathcal{M},\mathcal{M}',C)$ is the $\mathcal{R}$-triple of the rescalable integrable mixed twistor module $\nabla^{\mathrm{H}}_{\check{G}}(V)$, then we have an isomorphism $\mathcal{M} \cong \nabla^{\mathcal{R}}_{\check{G}}(V)$ of integrable $\mathcal{R}$-modules. Consequently, the irregular Hodge filtration of $\nabla^{\mathrm{H}}_{\check{G}}(V)$ is determined by ρ in the sense of Definition 3.5. ♢

It is easy to verify this conjecture in the examples from Section 5.1. As mentioned above, when assuming the conjecture, our main result Theorem 1.1 would be a rather direct consequence using a similar strategy to [CnDS21, Prop. 4.6] or to [CnDRS19, Thm. 5.9] (compare the expression (6.0.1) to the expression of the connection ∇ in [CnDS21, Lem. 4.3]), where an adapted basis for the irregular Hodge filtration was constructed. It is, however, unclear how to show the above conjecture when the irregular Hodge structure $\nabla^{\mathrm{H}}_{\check{G}}(V)$ is defined using the geometric construction from Section 2.3.

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