Universität Mannheim
Lehrstuhl für Mathematik VI

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Exercises Algebraic Geometry
Sheet 4 - solutions

1. **The tensor product and Hom:** We discuss here the relationship of the tensor product with the Hom-functor. We start with a property of the Hom-groups which is in some sense dual to the one for the tensor product we are interested in.

Lemma 1. *Let A be a ring and*

$$0 \to M'' \xrightarrow{\beta} M' \xrightarrow{\gamma} M \to 0$$

be a short exact sequence of A-modules (remember: this means β is injective, γ is surjective and $ker(\gamma) = im(\beta)$). Then for any A-module Q (in particular, for A itself) there is an exact sequence

$$0 \to Hom_A(M, Q) \xrightarrow{\gamma^*} Hom_A(M', Q) \xrightarrow{\beta^*} Hom_A(M'', Q)$$

(note that the β^* is not surjective in general). Conversely, a sequence

$$0 \to M'' \xrightarrow{\beta} M' \xrightarrow{\gamma} M \to 0$$

(this means that $\gamma \circ \beta = 0$ so $im(\beta) \subset ker(\gamma)$ but not $im(\beta) = ker(\gamma)$ in general) is exact if for all A-modules Q, we have that

$$0 \to Hom_A(M, Q) \xrightarrow{\gamma^*} Hom_A(M', Q) \xrightarrow{\beta^*} Hom_A(M'', Q) \tag{1}$$

is exact. Note that it follows from $\gamma \circ \beta = 0$ that $\beta^* \circ \gamma^* = 0$ (see the proof), so that it makes sense to call (1) a sequence.

Proof. First the definition of the maps γ^* and β^*: this is just the generalization of the dual map (hence the names): given $f \in Hom_A(M, Q)$ (resp. $g \in Hom_A(M', Q)$), we put $\gamma^*(f) := f \circ \gamma$ (resp. $\beta^*(g) := g \circ \beta$). It is easy to show that the maps defined in this way are homomorphisms of A-modules. Let us show that γ^* is injective: suppose that $\gamma^*(f)(a) = 0$ for all $a \in M'$, then f is necessarily zero in $Hom_A(M, Q)$, otherwise, given any $b \in M$ with $f(b) \neq 0$, we take a lift $\widetilde{a}$ of b to M', that is, any preimage under γ, and then $\gamma^*(f)(\widetilde{a}) \neq 0$. The next step is to show that $im(\gamma^*) = ker(\beta^*)$: First, the composition $\beta^* \circ \gamma^*$ is obviously zero because $\beta^* \circ \gamma^*(f) = f \circ \gamma \circ \beta = 0$ as already $\gamma \circ \beta = 0$. So let $g \in ker(\beta^*) \subset Hom_A(M', Q)$ be given, this means that $g \circ \beta$ is zero in $Hom_A(M'', Q)$. We will construct $f \in Hom_A(M, Q)$ with $\gamma^*(f) = g$: given $b \in M$, take any lift a as above (i.e. $\gamma(a) = b$) and **define** $f(b) := g(a)$. A priori, this may not be well-defined, it seems to depend on the choice of the lift $a \in \gamma^{-1}(b)$. But in fact it does not, suppose $\widetilde{a}$ to be another choice, then $a - \widetilde{a} \in ker(\gamma) = im(\beta)$ so that there is $c \in M''$ with $\beta(c) = a - \widetilde{a}$. Therefore $g(a - \widetilde{a}) = (g \circ \beta)(c) = 0$ by assumption. So we see that f is well-defined which shows that $g \in im(\beta^*)$.

For the other direction, note that given

$$M'' \xrightarrow{\beta} M' \xrightarrow{\gamma} M$$

with $\gamma \circ \beta = 0$, if γ is not surjective, then the canonical projection $\pi : M \to M/im(\gamma) = coker(\gamma)$ is an element in $Hom_A(M, coker(\gamma))$, different from zero, but the composition $\gamma^*(\pi) = \pi \circ \gamma$ is zero in $Hom_A(M', coker(\gamma))$. So there is an A-module $Q := coker(\gamma)$ such that $Hom_A(M, Q) \xrightarrow{\gamma^*} Hom_A(M', Q)$ is no injective. Similarly, let $\psi : M' \to M'/im(\beta) = coker(\beta)$ be the canonical projection, this is an element in $ker(\beta^*) \subset Hom_A(M', Q)$ (because $\beta^*(\psi) = \psi \circ \beta = 0$) where $Q := coker(\beta)$. If $im(\beta) \subsetneq ker(\gamma)$ then there cannot be any $\Psi \in Hom_A(M, Q)$ with $\psi = \Psi \circ \gamma$: If there were such an Ψ, take any non-zero element $x \in ker(\gamma) \backslash im(\beta)$, then $\psi(x) \neq 0$, but $\gamma(x) = 0$ so that $\psi \circ \gamma(x) = 0$, a contradiction. This finishes the proof. $\square$

The next step is a reinterpretation of the tensor product in terms of the *Hom*-groups.

Lemma 2. *Let M, N, Q A-modules, then there are natural isomorphisms*

$$Bilin_A(M \times N, Q) \cong Hom_A(M \otimes_A N, Q) \quad ; \quad Bilin_A(M \times N, Q) \cong Hom_A(M, Hom_A(N, Q))$$

where $Bilin_A(M \times N, Q)$ is A-module of bilinear maps from $M \times N$ to Q.

Proof. The first isomorphism is the universal property of the tensor product, namely it is given by sending a bilinear map ψ to the map defined by $m \otimes n \mapsto \psi(m, n)$. The second isomorphism is also obvious: given $\psi \in Bilin_A(M \times N, Q)$, we have that for any $m \in M$, the map $\psi_m : N \to Q$ sending $n \in N$ to $\psi_m(n) := \psi(m, n) \in Q$ is A-linear, i.e., $\psi_m \in Hom_A(N, Q)$, and that the map sending $m \to \psi_m$ is also A-linear. This means that ψ gives an element in $Hom_A(M, Hom_A(N, Q))$ and one checks that both modules are isomorphic under this map (namely, construct the obvious inverse). $\square$

We can now use these two lemmas to prove the following result.

Corollary 1. *Let the above exact sequence of A-modules be given, then for any A-module P, there is an exact sequence*

$$M'' \otimes_A P \overset{\beta \otimes id}{\longrightarrow} M' \otimes_A P \overset{\gamma \otimes id}{\longrightarrow} M \otimes_A P \longrightarrow 0 \tag{2}$$

Note that this does not mean that $\beta \otimes id$ is injective.

Proof. The second part of the first lemma shows that it is sufficient to show that

$$0 \longrightarrow Hom_A(M \otimes_A P, N) \overset{(\gamma \otimes id)^*}{\longrightarrow} Hom_A(M' \otimes_A P, N) \overset{(\beta \otimes id)^*}{\longrightarrow} Hom_A(M'' \otimes_A P, N)$$

is exact for any A-module N. By the second lemma, we know that this is equivalent to

$$0 \longrightarrow Hom_A(M, Hom_A(P, N)) \overset{(\gamma \otimes id)^*}{\longrightarrow} Hom_A(M', Hom_A(P, N)) \overset{(\beta \otimes id)^*}{\longrightarrow} Hom_A(M'', Hom_A(P, N))$$

but that this sequence is exact follows simply by applying the first part of the first lemma for $Q := Hom_A(P, N)$. $\square$

2. **Exterior product and the Grassmann varieties:** We will use the exterior product to determine (in a special case) the image of the Plücker embedding. This will show that the Grassmannian is a projective variety. We start with a definition

Definition 0.1. *Let A be a ring (we will use here only the case where A is a field) and M an A-module. Then we define*

$$\bigwedge_A^p M := \left(\underbrace{M \otimes_A \ldots \otimes_A M}_{p \text{ times}} \right) / N$$

where N is the submodule generated by all elements of the form $m_1 \otimes \ldots \otimes m_p$ such that there is $i \neq j$ with $m_i = m_j$. Denote by $x_1 \wedge \ldots \wedge x_p \in \bigwedge^p M$ the class of $x_1 \otimes \ldots \otimes x_n$. There is a homomorphism of A-modules

$$\begin{array}{ccc} \bigwedge^p M \otimes_A \bigwedge^q M & \longrightarrow & \bigwedge^{p+q} M \\ (x_1 \wedge \ldots \wedge x_p) \otimes (y_1 \wedge \ldots \wedge y_q) & \longmapsto & x_1 \wedge \ldots \wedge x_p \wedge y_1 \wedge \ldots \wedge y_q \end{array}$$

which satisfies $\underline{x} \wedge \underline{y} = (-1)^{pq} \underline{y} \wedge \underline{x}$ for all $\underline{x} \in \bigwedge^p M$ and $\underline{y} \in \bigwedge^q M$ (This can be expressed by saying that $\bigwedge^\bullet M := \oplus_{p \geq 0} \bigwedge^p M$ is a graded-commutative algebra, note that similarly $\oplus_{p \geq 0} \otimes^p M$ is a non-commutative algebra). In particular, $x \wedge y = -y \wedge x$ for any $x, y \in M$, this can of course already be seen from the very definition of the exterior product.

Lemma 3. *(a) Let M be a finitely generated free A-module, i.e., there are elements $m_1, \ldots, m_k$ such that*

$$\begin{array}{ccc} A^k \cong \oplus A e_i & \to & M \\ e_i & \mapsto & m_i \end{array}$$

is an isomorphism (here e_i denots the ith standard basis vector of A^k). Then

$$\bigwedge^p M \overset{\cong}{\longrightarrow} A^{\binom{k}{p}} \cong \oplus_{1 \leq i_1 < \ldots i_p \leq k} A \, e_{i_1} \wedge \ldots \wedge e_{i_p}$$

We denote the vector $e_{i_1} \wedge \ldots \wedge e_{i_p} \in \bigwedge^p M$ by $e_{i_1, \ldots, i_p}$. In particular, $\bigwedge^k M$ is free of rank one (with generator $e_{1,2,\ldots,n} = e_1 \wedge \ldots \wedge e_n$), i.e., isomorphic to A and $\bigwedge^p M = 0$ for all $p > k$.

(b) For any $f \in Hom_A(M,N)$ and any $p \in \mathbb{N}$ there is an induced homomorphism $\bigwedge^p f : \bigwedge^p M \to \bigwedge^p N$. Let $M \cong \oplus_{i=1}^k Ae_i$, $N \cong \oplus_{j=1}^l A\widetilde{e}_j$ and $(F_{ji}) \in M(l \times k, A)$ the matrix of f with respect to $\underline{e}$ and $\underline{\widetilde{e}}$ (i.e., $f(e_i) = \sum_{j=1}^l F_{ji}\widetilde{e}_j$). Then the matrix of $\bigwedge^p f$ with respect to $(\underline{e}_{i_1,\ldots,i_p})$ and $(\underline{\widetilde{e}}_{j_1,\ldots,j_p})$ is given by

$$(\bigwedge^p f)(\underline{e}_{i_1,\ldots,i_p}) = \sum_{1\leq j_1<\ldots<i_p\leq l} (F^{j_1,\ldots,j_p}_{i_1,\ldots,i_p}) \cdot \underline{\widetilde{e}}_{j_1,\ldots,j_p}$$

where $F^{j_1,\ldots,j_p}_{i_1,\ldots,i_p}$ is the determinant of the submatrix of (F_{ji}) consisting of the $i_1,\ldots,i_p$-th column and the $j_1,\ldots,j_p$-th row.

Proof. The first part simply follows from the properties of the exterior product, i.e.,

$$\left(\sum_{i=1}^n \lambda_i e_i\right) \wedge \left(\sum_{i=1}^n \mu_i e_i\right) = \sum_{i<j} (\lambda_i\mu_j - \mu_i\lambda_j) e_i \wedge e_j = \sum_{i<j} \begin{vmatrix} \lambda_i & \mu_i \\ \lambda_j & \mu_j \end{vmatrix} e_i \wedge e_j$$

and similar for the higher exterior products. A similar calculation shows the second part (i.e., the last formula is already the proof of (b) for the case $p=2$, here (F_{ji}) would be the matrix $\begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \mu_1 & \mu_2 & \ldots & \mu_n \end{pmatrix}$. □

This lemma allows us to rewrite the Plücker embedding as follows (in the sequel the ring A from above is the field k).

$$\begin{array}{rcl} \mathrm{Pl} : \mathrm{Gr}(l,n) & \longrightarrow & \mathbb{P}(\bigwedge^l k^n) \cong \mathbb{P}^{\binom{n}{l}-1} \\ L := \mathrm{span}\langle v_1,\ldots,v_l\rangle = \mathrm{span}\langle \underline{e}\cdot A\rangle & \longmapsto & v_1 \wedge \ldots \wedge v_l = (\sum_{i=1}^n A_{i1}e_i) \wedge \ldots \wedge (\sum_{i=1}^n A_{il}e_i) \\ & & = \sum_{1\leq i_1<\ldots<i_l\leq n} A_{i_1,\ldots,i_l} e_{i_1} \wedge \ldots \wedge e_{i_l} \end{array}$$

Note that in order to link the notation to that of the last lemma, we can write $L := im(A)$, where $A : k^l \to k^n$ is defined as $A(e_i) = \sum_{j=1}^n A_{ji}e_j$ for all $i \in \{1,\ldots,l\}$. Then the minor $A_{i_1,\ldots,i_l}$ is (in the notation from above) $A^{i_1,\ldots,i_l}_{1,\ldots,l}$.

The homogenous coordinates on $\mathbb{P}^{\binom{n}{l}-1}$ are written as $(\lambda_{1,\ldots,1} : \lambda_{2,1\ldots,1} : \ldots : \lambda_{i_1,\ldots,i_l} : \ldots)$ so that Pl can be expressed by saying that $\lambda_{i_1\ldots i_l} := A_{i_1\ldots i_l}$. We see that any $w \in \mathbb{P}(\bigwedge^l k^n)$ is in the image of the Plücker map iff there are vectors $v_1,\ldots v_n \in k^n$ such that $w = v_1 \wedge \ldots \wedge v_n$. Let us now specify to the case $l=2$.

Lemma 4. *A vector $w \in \bigwedge^2 k^n$ lies in the image of the Plücker embedding if and only if $w \wedge w = 0$ in $\bigwedge^4 k^n$.*

Proof. We give an elementary proof: Let $w = \sum_{1\leq i<j\leq n} \lambda_{ij}e_{ij}$, then

$$w \wedge w = \sum_{1\leq i<j<k<l\leq n} (\lambda_{ij}\lambda_{kl} - \lambda_{ik}\lambda_{jl} + \lambda_{il}\lambda_{jk})\, e_{ijkl}$$

which we write as

$$w \wedge w = (\lambda_{12}\lambda_{34} - \lambda_{13}\lambda_{24} + \lambda_{14}\lambda_{23})\, e_{1234} + \sum_{1\leq i<j<k<l; l>4} (\lambda_{ij}\lambda_{kl} - \lambda_{ik}\lambda_{jl} + \lambda_{il}\lambda_{jk})\, e_{ijkl}$$

We can suppose that at least one of the coefficients $\lambda_{12}, \lambda_{34}, \lambda_{13}, \lambda_{24}, \lambda_{14}\lambda_{23}$ is non-zero, if not, we have reduced the length of the sum representing w by one and we continue inductively. So let $\lambda_{14} \neq 0$. We set $e_1^{(1)} := \lambda_{14}e_1 + \lambda_{24}e_2 + \lambda_{34}e_3$, $e_i^{(1)} := e_i$ for all $i > 1$, and we obtain new coefficients $\lambda_{ij}^{(1)}$ defined by

$$w = \sum_{i<j} \lambda_{ij}^{(1)} e_i^{(1)} \wedge e_j^{(1)}$$

The choice of $e_1^{(1)}$ implies that $\lambda_{24}^{(1)} = \lambda_{34}^{(1)} = 0$ and $\lambda_{14}^{(1)} = 1$. But the equation $w \wedge w = 0$ also gives $\lambda_{12}^{(1)}\lambda_{34}^{(1)} - \lambda_{13}^{(1)}\lambda_{24}^{(1)} + \lambda_{14}^{(1)}\lambda_{23}^{(1)} = 0$, so that $\lambda_{23}^{(1)} = 0$. This shows that we can write w as

$$w = e_1^{(1)} \wedge \left(\lambda_{12}^{(1)}e_2^{(1)} + \lambda_{13}^{(1)}e_3^{(1)} + \lambda_{14}^{(1)}e_4^{(1)}\right) + \sum_{i<j, j>4} \lambda_{ij}^{(1)} e_i^{(1)} \wedge e_j^{(1)}$$

Now put: $e_1^{(2)} := e_1^{(1)}$, $e_2^{(2)} := \lambda_{12}^{(1)} e_2^{(1)} + \lambda_{13}^{(1)} e_3^{(1)} + \lambda_{14}^{(1)} e_4^{(1)}$, $e_i^{(2)} := e_i^{(1)}$ for all $i > 2$ and, as before, define the new coefficients by

$$w = \sum_{i<j} \lambda_{ij}^{(2)} e_i^{(2)} \wedge e_j^{(2)}$$

so that

$$w = e_1^{(2)} \wedge e_2^{(2)} + \sum_{i<j; j>4} \lambda_{ij}^{(2)} e_i^{(2)} \wedge e_j^{(2)} \tag{3}$$

We continue the expansion as

$$w = e_1^{(2)} \wedge e_2^{(2)} + \lambda_{15}^{(2)} e_1^{(2)} \wedge e_5^{(2)} + \lambda_{25}^{(2)} e_2^{(2)} \wedge e_5^{(2)} + \lambda_{35}^{(2)} e_3^{(2)} \wedge e_5^{(2)} + \lambda_{45}^{(2)} e_4^{(2)} \wedge e_5^{(2)} + \sum_{i<j; j>5} \lambda_{ij}^{(2)} e_i^{(2)} \wedge e_j^{(2)}$$

On the other hand, the equation $w \wedge w = 0$ also gives

$$\lambda_{12}^{(2)} \lambda_{35}^{(2)} - \lambda_{13}^{(2)} \lambda_{25}^{(2)} + \lambda_{15}^{(2)} \lambda_{23}^{(2)} = 0 \quad ; \quad \lambda_{12}^{(2)} \lambda_{45}^{(2)} - \lambda_{14}^{(2)} \lambda_{25}^{(2)} + \lambda_{15}^{(2)} \lambda_{24}^{(2)} = 0$$

but as $\lambda_{13}^{(2)} = \lambda_{23}^{(2)} = \lambda_{14}^{(2)} = \lambda_{24}^{(2)} = 0$ and $\lambda_{12}^{(2)} = 1$, we obtain $\lambda_{35}^{(2)} = \lambda_{45}^{(2)} = 0$ so that

$$\begin{aligned} w &= e_1^{(2)} \wedge e_2^{(2)} + \lambda_{15}^{(2)} e_1^{(2)} \wedge e_5^{(2)} + \lambda_{25}^{(2)} e_2^{(2)} \wedge e_5^{(2)} + \textstyle\sum_{i<j; j>5} \lambda_{ij}^{(2)} e_i^{(2)} \wedge e_j^{(2)} \\ &= (\underbrace{e_1^{(2)} - \alpha_{25}^{(2)} e_5^{(2)}}_{e_1^{(3)}}) \wedge (\underbrace{e_2^{(2)} + \alpha_{15}^{(2)} e_5^{(2)}}_{e_2^{(3)}}) + \sum_{i<j; j>5} \lambda_{ij}^{(2)} e_i^{(2)} \wedge e_j^{(2)} \end{aligned}$$

so that by putting $e_i^{(3)} = e_i^{(2)}$ for all $i > 2$, we obtain the same expression as equation (3), but with the index $j > 5$, so that by induction, we can finish the proof. $\square$

Using this lemma, we can write down equations for the image of Pl, namely, with $w = \sum_{1 \le i < j \le n} \lambda_{ij} e_{ij} \in \bigwedge^2 k^n$, we have, as already used in the proof, that

$$w \wedge w = \sum_{1 \le i < j < k < l \le n} (\lambda_{ij}\lambda_{kl} - \lambda_{ik}\lambda_{jl} + \lambda_{il}\lambda_{jk})\, e_{ijkl}$$

This shows that

$$\mathrm{Im}(\mathrm{Pl}) = V\left(\lambda_{ij}\lambda_{kl} - \lambda_{ik}\lambda_{jl} + \lambda_{il}\lambda_{jk}\right)_{1 \le i < j < k < l \le n}$$

which is a closed subset of $\mathbb{P}^{\binom{n}{2}-1}$. In particular, for $n = 4$ we have a single equation $\mathrm{Gr}(2,4) \cong V(\lambda_{12}\lambda_{34} - \lambda_{13}\lambda_{24} + \lambda_{14}\lambda_{23}$, where $(\lambda_{12} : \lambda_{13} : \lambda_{14} : \lambda_{23} : \lambda_{24} : \lambda_{34})$ are the homogenous coordinates of $\mathbb{P}^5$.

The following considerations show that the canonical affine cover of $\mathrm{Im}(\mathrm{Pl} : \mathrm{Gr}(2,n) \to \mathbb{P}^{\binom{n}{2}-1})$ with respect to the above coordinates can be simply described in terms of the matrices representing the point of $\mathrm{Gr}(2,n)$. Let

$$L = span \left\langle (\underline{e}) \cdot \begin{pmatrix} a_1 & \dots & a_i & \dots & a_j & \dots & a_n \\ b_1 & \dots & b_i & \dots & b_j & \dots & b_n \end{pmatrix}^{tr} \right\rangle \in \mathrm{Gr}(2,n)$$

and suppose that $a_i \neq 0$ and that $b_j - b_i \frac{a_j}{a_i} \neq 0$, then we may write

$$L = span \left\langle (\underline{e}) \cdot \begin{pmatrix} \widetilde{a}_1 & \dots & 1 & \dots & 0 & \dots & \widetilde{a}_n \\ \widetilde{b}_1 & \dots & 0 & \dots & 1 & \dots & \widetilde{b}_n \end{pmatrix}^{tr} \right\rangle \in \mathrm{Gr}(2,n)$$

It is easy to see that the Plücker coordinate x_{ij} is equal to one, i.e, the subset

$$\left\{ L \in \mathrm{Gr}(2,n) \,|\, L = span \left\langle (\underline{e}) \cdot \begin{pmatrix} \widetilde{a}_1 & \dots & 1 & \dots & 0 & \dots & \widetilde{a}_n \\ \widetilde{b}_1 & \dots & 0 & \dots & 1 & \dots & \widetilde{b}_n \end{pmatrix}^{tr} \right\rangle \right\}$$

is precisely the affine chart $U_{ij} := \mathrm{Gr}(2,n) \cap \{x_{ij} \neq 0\}$. As Pl is an isomorphism, we see that $\mathrm{Pl}^{-1}(U_{ij}) \cong k^{2(n-1)} = \left\{ (\widetilde{a}_k, \widetilde{b}_l)_{k,l \neq i,j} \right\}$.