Multiscale Analysis and Localization

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Outline

This is about the proof of localization for random operators using multiscale analysis.

- The metal-insulator transition: physics and history.
- Random Schrödinger Operators.
- The Multiscale Induction.
- Recent results.

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The metal-insulator transition: physics and history.

Starting point is the Schrödinger equation

$$\dot{\psi}(t) = -iH\psi(t), \psi(0) = \psi_0,$$

where

- $\psi : \mathbb{R} \to \mathcal{H}$ describes the time evolution of a state in the Hilbert space $\mathcal{H}$
- and H is a self-adjoint operator.

Typical:

- $\mathcal{H} = L^2(\mathbb{R}^3)$,
- $H = -\Delta + V$, $-\Delta$ - kinetic energy, V - potential energy

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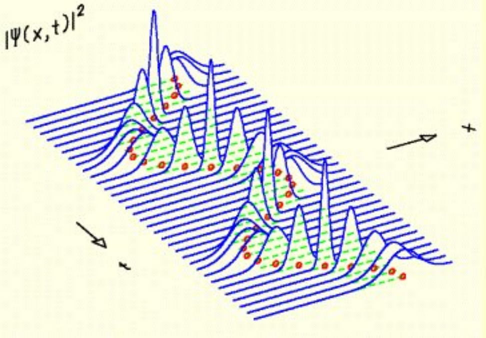

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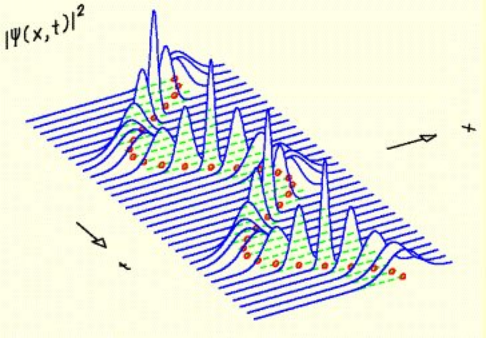

The metal-insulator transition: physics and history.

Easy: the solution of the Schrödinger equation is

$$\psi(t) = e^{-iHt}\psi_0.$$

But: what does it look like?
To this end one studies the spectral resolution of H.

- If ψ_0 is an eigenvector of H with eigenvalue $E_0 \Rightarrow$

$$\psi(t) = e^{-iE_0 t}\psi_0.$$

  One speaks of a bound state.

- If the spectral measure $\rho^H_{\psi_0}$ of ψ_0 w.r.t H is continuous,

$$\Rightarrow \lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T} \|\chi_B \psi(t)\|^2 dt = 0 \text{ for compact } B.$$

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- Atomic models are described by a negative potential V that decays rapidely enough near infinity:one gets bound states for negative energies and scattering states for positive energies.
- Solid states with perfect order are described by a periodic V: $\Rightarrow H = -\Delta + V$ has only scattering states.
- P.W. Anderson proposed a new paradigm for disordered solids in dimension ≥ 3:

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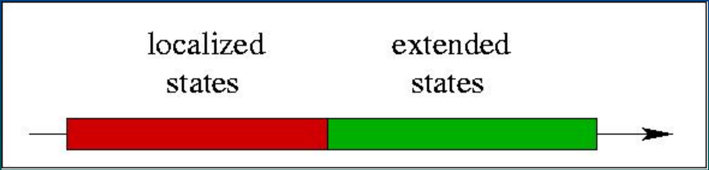

Localization

Once translated into the language of spectral theory there is a transition from a
localized phase that exhibits pure point spectrum
(= only bound states = no transport)
to a
delocalized phase with absolutely continuous spectrum
(= scattering states = transport)
However, for genuine random models, there is no rigorous proof of the existence of a transition or even of the appearance of spectral components other than pure point, so far.
This is a quite strange situation: the unperturbed problem exhibits extended states and purely a.c. spectrum but for the perturbed one can prove the opposite spectral behavior only.

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Disorder?

Disorder is modelled by random potentials:

$$H(\omega) = -\Delta + V_\omega$$

E.g., $\Omega = \mathbb{R}^{\mathbb{Z}^d}$, $\mathbb{P} = \mu^{\mathbb{Z}^d}$ a probability space describing independent, identically distributed coupling constants

$$V_\omega = \sum_{k\in\mathbb{Z}^d} \omega_k \cdot v(\cdot - k)$$

sometimes called alloy type models. Localization has been proven for such models under additional technical hypotheses on μ and v.

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- new techniques
- led to new aspects of spectral theory ...
- ... with generic exotic spectral types

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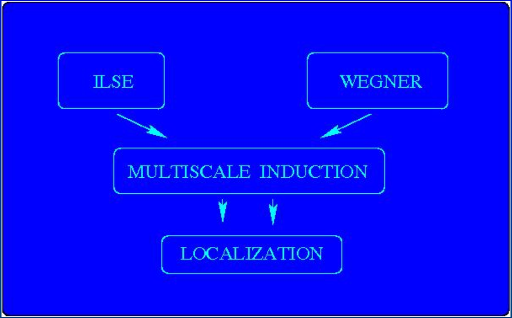
Multiscale induction



Induction on length ℓ of boxes $\Lambda = \Lambda_\ell(x)$; analysis of box hamiltonians $H_\Lambda(\omega)$.

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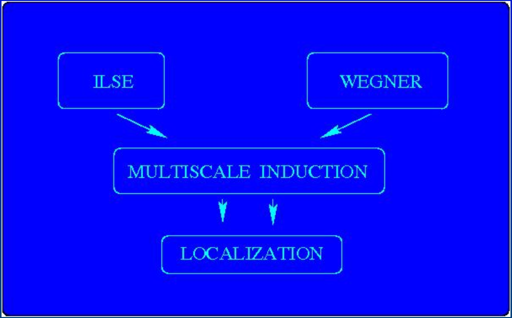

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- Good news: $H_\Lambda(\omega)$ is elliptic can be expanded in eigenfunctions.
- Bad news: Spectral type of $H_\Lambda(\omega)$ does not determine the spectral type of $H(\omega)$.
- Way out: decay properties of the resolvent

$$(H_\Lambda(\omega) - E)^{-1}.$$

In fact, suppose for $I \subset \mathbb{R}$, $E \in I$ and $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\|\chi_x(H_\Lambda(\omega) - E)^{-1}\chi_y\| \leq C \cdot \exp(-m \cdot |x-y|).$$

for some $m > 0$ and $|x-y|$ large enough.
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