Localization and delocalization for nonstationary models

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Overview

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This is work from the joint project with W. Kirsch

References

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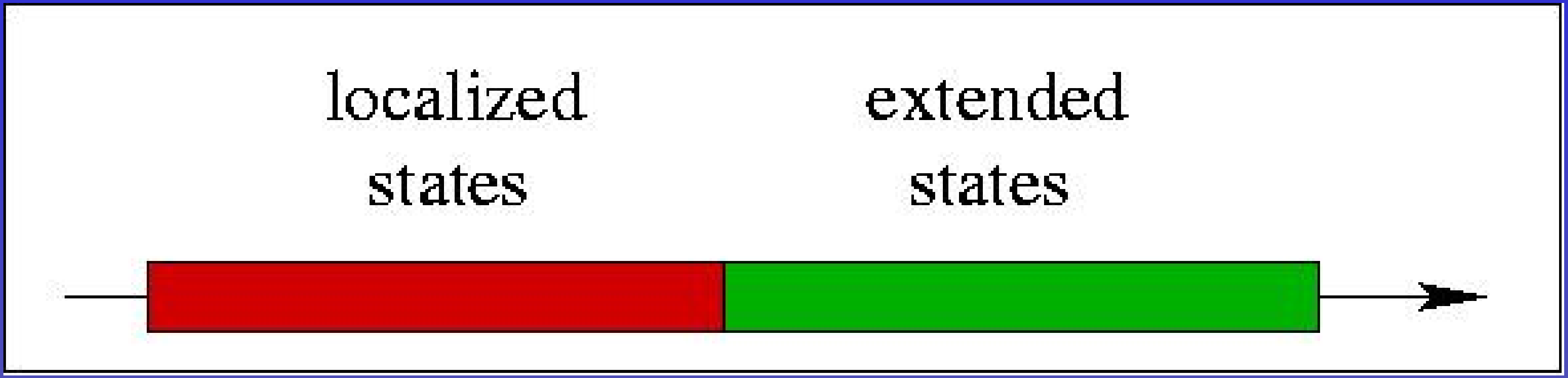

Figure 1: Metal insulator transition

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Quite recently, an order parameter has been introduced by Germinet and Klein to characterize the range of energies where a multiscale scenario provides a proof of a localized regime. Here the important parameter is the energy. However, so far there is no rigorous proof of the existence of a transition or even of the appearance of spectral components other than pure point.

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This is a quite strange situation: the unperturbed problem exhibits extended states but for the perturbed one can prove the other spectral type only.

overview

Sparse Potentials

As a typical example let us consider the following model in $L^2(\mathbb{R}^d)$,

$$H(\omega) = -\Delta + V_\omega, \text{ where } V_\omega(x) = \sum_{k \in \mathbb{Z}^m} \xi_k(\omega) f(x-k),$$

where $f \leq 0$ is a compactly supported single site potential and the ξ_k are independent Bernoulli variables with $p_k := \mathbb{P}\{\xi_k = 1\}$.

Figure 2: A typical random sparse potential: the peaks are at lattice points k where $\xi_k = 1$

To understand the appearance of a metallic regime, we recall the following facts from scattering theory:

Cook's criterion

We write $-\Delta = H_0$ so that the operators we are interested in can be written as $H = H_0 + V$. By $\sigma_{ac}(H)$ we denote the absolutely continuous spectrum, related to delocalized states.

Theorem 1. *(Cooks criterion)*
If for some $T_0 > 0$ and all ϕ in a dense set

$$\int_{T_0}^{\infty} \|Ve^{-itH_0}\phi\| dt < \infty \qquad (*)$$

then $\Omega_- := lim_{t\to\infty} e^{itH} e^{-itH_0}$ exists and, consequently, $[0, \infty) \subset \sigma_{ac}(H)$, i.e., there are scattering states for H and any nonnegative energy.

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However, the following nice result holds; see Hundertmark and Kirsch [12] who also provided the absolutely correct name:
Theorem 2. *(Almost surely free lunch theorem)*
Assume that

$$W(x) := \left(\mathbb{E}(V_\omega(x)^2)\right)^{\frac{1}{2}} \overset{!}{\leq} C(1+|x|)^{-(1+\epsilon)}.$$

Then V_ω satisfies Cook's criterion for a.e. ω.

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If we fix $d \geq 3$ and $\frac{d}{2} + \frac{1}{2} < \alpha < d$ and $p_k \sim k^{-\alpha}$ we can moreover control the essential spectrum below 0. We can summarize this in the following picture:

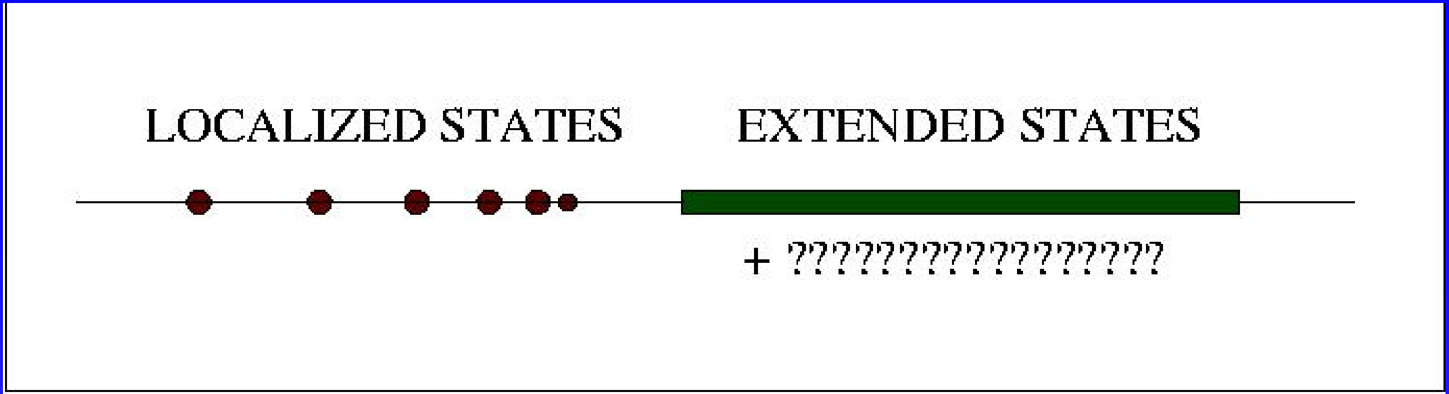

Figure 3: Conclusion and open problems for the sparse model

overview

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$$= \int_{T_0}^{\infty} \|W(x)e^{-itH_0}\phi\| dt$$

$\square$

back

Continuum surface models

Consider the following self-adjoint random operator in $L^2(\mathbb{R}^d)$,

$$H(\omega) = -\Delta + V_\omega, \text{ where } V_\omega(x) = \sum_{k \in \mathbb{Z}^m} q_k(\omega) f(x - (k, 0)),$$

the q_k are i.i.d. random variables and $f \geq 0$ is a single site potential that satisfy certain technical assumptions.

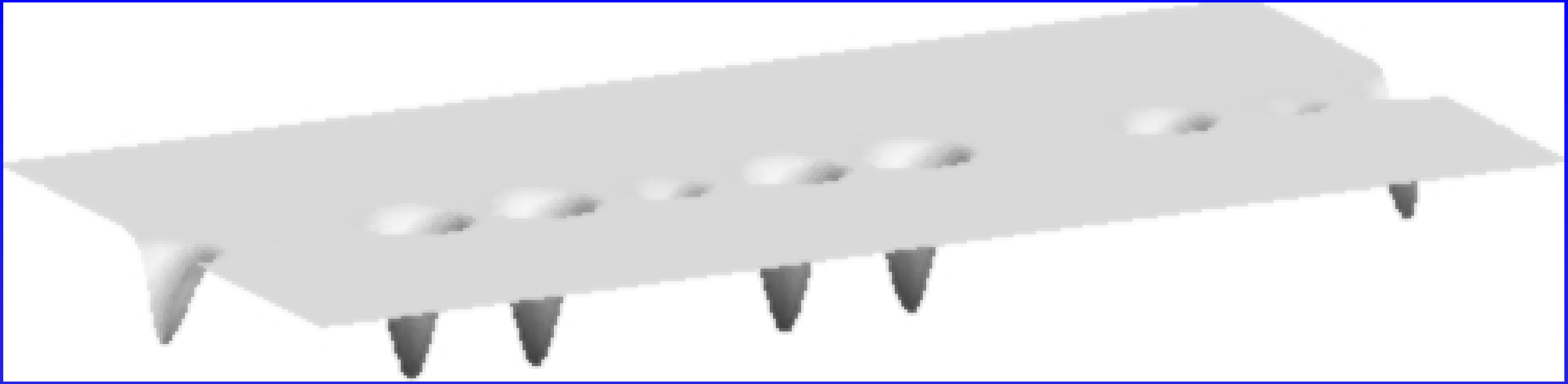

Figure 4: A typical random surface potential

The spectrum

It is not hard to see that

$$\sigma(H(\omega)) = [E_0, \infty) \text{ where } E_0 = \inf \sigma(-\Delta + q_{\mathrm{min}} \cdot f^{\mathrm{per}}),$$

and

$$f^{\mathrm{per}} = \sum_{k \in \mathbb{Z}^m} f(x - (k, 0))$$

denotes the periodic continuation of f along the surface.

Near the bottom of the spectrum E_0 one expects localization, i.e. suppression of transport as is typical for insulators. This is the content of the main result in [3]. For nonnegative energies one expects extended states.

Extended states

To stress the existence of a metallic phase let us cite Theorem 4.3 of [12]:

Theorem 3. *Let* $H(\omega)$ *satisfy the assumptions. Then we have, for every* $\omega \in \Omega$*:*

$$[0, \infty) \subset \sigma_{\mathrm{ac}}(H(\omega)).$$

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→Summary

The idea of the *Proof* is that a wave packet with velocity pointing away from the surface will escape the influence of the surface potential and is asymptotically free. The rigorous implementation of this idea uses Enss' technique from scattering theory.

Localized states

Theorem 4. *Let* $H(\omega)$ *be as in the above assumptions with* $\tau > d/2$.

(a) *There exists an* $\varepsilon > 0$ *such that in* $[E_0, E_0 + \varepsilon]$ *the spectrum of* $H(\omega)$ *is pure point for almost every* $\omega \in \Omega$, *with exponentially decaying eigenfunctions.*

(b) *Assume that* $p < 2(2\tau - m)$. *Then there exists an* $\varepsilon > 0$ *such that in* $[E_0, E_0 + \varepsilon] = I$ *we have strong dynamical localization in the sense that:*

$$\mathbb{E}\{\sup_{t>0} \| |X|^p e^{-itH(\omega)} P_I(H(\omega)) \chi_K \|\} < \infty$$

for every compact set $K \subset \mathbb{R}^d$.

A consequence is pure point spectrum in the interval $[E_0, E_0 + \varepsilon] = I$. Together with the result on extended states we get the following picture that leaves open some important questions.

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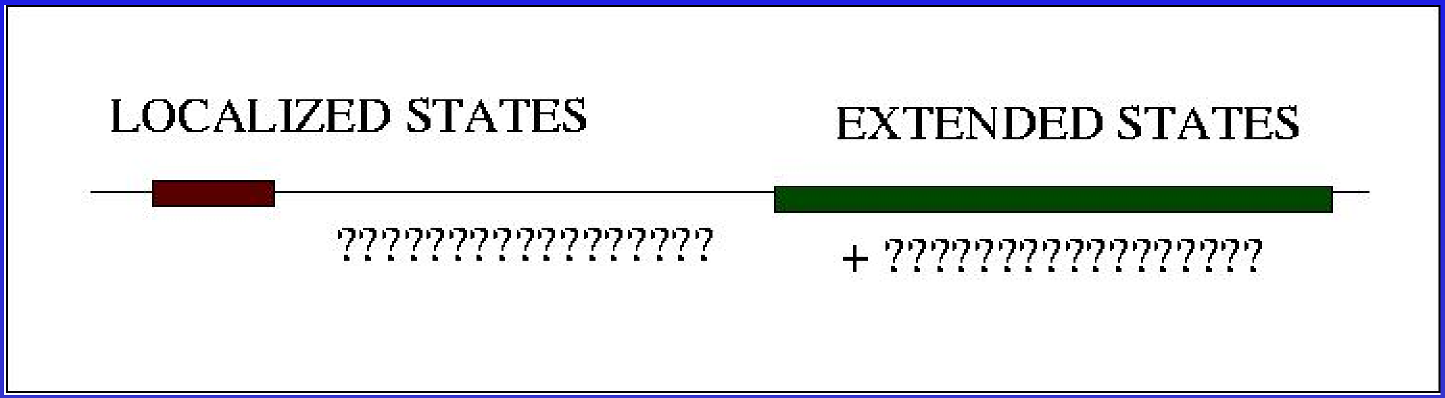

Figure 5: Conclusion and open problems for the continuum surface model

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References

[1] P.W. Anderson: Absence of diffusion in certain random lattices. *Phys. Rev.* **109**, 1492–1505 (1958)

[2] J. Bourgain, M. Goldstein: On nonperturbative localization with quasiperiodic potential. *Ann. of Math.*, **152**, 835–879 (2000).

[3] A. Boutet de Monvel, P. Stollmann: Dynamical localization for continuum random surface models, *Arch. Math.*, to appear

[4] A.Boutet de Monvel, A. Surkova: Localisation des états de

surface pour une classe d'opérateurs de Schrödinger discrets à potentiels de surface quasi-périodiques, *Helv. Phys. Acta* **71**(5), 459–490 (1998).

[5] V. Grinshpun: Localization for random potentials supported on a subspace, *Lett. Math. Phys.* **34**(2), 103–117 (1995).

[6] V. Jakšić, Y. Last: Corrugated surfaces and a.c. spectrum, *Rev. Math. Phys.* **12**(11), 1465–1503 (2000).

[7] V. Jakšić, S. Molchanov: On the surface spectrum in dimension two, *Helv. Phys. Acta* **71**(6), 629–657 (1998).

[8] V. Jakšić, S. Molchanov: On the spectrum of the surface Maryland model, *Lett. Math. Phys.* **45**(3), 189–193 (1998).

[9] V. Jakšić, S. Molchanov: Localization of surface spectra, *Comm. Math. Phys.* **208**(1), 153–172 (1999).

[10] V. Jakšić, S. Molchanov, L. Pastur: On the propagation properties of surface waves, in “Wave propagation in complex media (Minneapolis, MN, 1994),” IMA Math. Appl., Vol. **96**, Springer, New York, 1998, pp. 143–154.

[11] S.Jitomirskaya: Metal-Insulator transition for the Almost Mathieu operator. *Annals of Math.*, **150**, 1159–1175 (1999)

[12] D. Hundertmark, W. Kirsch: Spectral theory of sparse potentials, in “Stochastic processes, physics and geometry: new interplays, I (Leipzig, 1999),” Amer. Math. Soc., Providence, RI, 2000, pp. 213–238.

[13] W. Kirsch, P. Stollmann, G. Stolz: Localization for random perturbations of periodic Schrödinger operators, *Random Oper. Stochastic Equations* **6**(3), 241–268 (1998).

[14] W. Kirsch, P. Stollmann, G. Stolz: Anderson localization for random Schrödinger operators with long range Interactions, *Comm. Math. Phys.* **195**(3), 495–507 (1998).

[15] P. Stollmann: *Caught by disorder – bound states in random media*, Progress in Math. Phys., vol. 20, Birkhäuser, Boston, 2001

[16] P. Stollmann: Wegner estimates and localization for continuum Anderson models with some singular distributions, *Arch. Math.* **75**(4), 307–311 (2000).

The almost Mathieu operator

The underlying Hilbert space is $l^2(\mathbb{Z})$. Consider parameters $\alpha, \lambda, \theta \in \mathbb{R}$ and define the selfadjoint, bounded operator $h_{\alpha,\lambda,\theta}$ by

$$(h_{\alpha,\lambda,\theta}u)(n) = u(n+1) + u(n-1) + \lambda \cos(2\pi(\alpha n + \theta))u(n),$$

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Note that this operator is a discrete Schrödinger operator with a potential term with the coupling constant λ in front and the discrete analog of the Laplacian. For irrational α the potential term is an almost periodic function on $\mathbb{Z}$. Basically, there is a metal insulator transition at the critical value 2 for the coupling constant λ. This is work of Jitomirskaya [11]; see also recent contributions by Bourgain and collaborators, [2] back

Assumptions

(1) $0 < m < d$ and points in $\mathbb{R}^d = \mathbb{R}^m \times \mathbb{R}^{d-m}$ are written as pairs, if convenient;

(2) The single site potential $f \geq 0$, $f \in L^p(\mathbb{R}^d)$ where $p \geq 2$ if $d \leq 3$ and $p > d/2$ if $d > 3$, and $f \geq \sigma > 0$ on some open set $U \neq \emptyset$ for some $\sigma > 0$.

(3) The q_k are i.i.d. random variables distributed with respect to a probability measure μ on $\mathbb{R}$, such that $\operatorname{supp} \mu = [q_{\mathsf{min}}, 0]$ with $q_{\mathsf{min}} < 0$.

We will sometimes need further assumptions on the single site distribution μ:

(4) μ is *Hölder continuous*, i.e. there are constants $C, \alpha > 0$ such that

$$\mu[a, b] \leq C(b-a)^{\alpha} \text{ for } q_{\mathsf{min}} \leq a \leq b \leq 0.$$

(5) *Disorder assumption*: there exist $C, \tau > 0$ such that

$$\mu[q_{\mathsf{min}}, q_{\mathsf{min}} + \varepsilon] \leq C \cdot \varepsilon^{\tau} \text{ for } \varepsilon > 0.$$

back to the model
back to localization
back to extended states