Spectral properties of corrugated surfaces or Localization and delocalization for nonstationary models

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Overview

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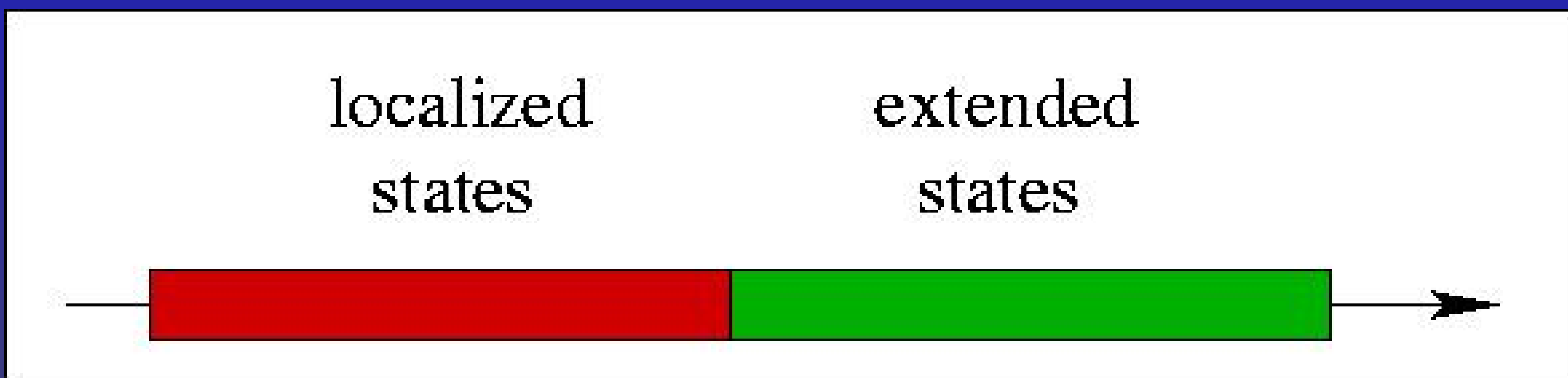

Figure 1: Metal insulator transition

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An instance where something like that has been verified rigorously is supplied by the **almost Mathieu operator**, a model with modest disorder for which the parameter that triggers the transition is the strength of the coupling.

The almost Mathieu operator

The underlying Hilbert space is $l^2(\mathbb{Z})$. Consider parameters $\alpha, \lambda, \theta \in \mathbb{R}$ and define the selfadjoint, bounded operator $h_{\alpha,\lambda,\theta}$ by

$$(h_{\alpha,\lambda,\theta}u)(n) = u(n+1) + u(n-1) + \lambda \cos(2\pi(\alpha n + \theta))u(n),$$

for $u = (u(n))_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$.

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Note that this operator is a discrete Schrödinger operator with a potential term with the coupling constant λ in front and the discrete analog of the Laplacian. For irrational α the potential term is an almost periodic function on $\mathbb{Z}$. Basically, there is a metal insulator transition at the critical value 2 for the coupling constant λ. As references, we mention

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Quite recently, an order parameter has been introduced by Germinet and Klein to characterize the range of energies where a multiscale scenario provides a proof of a localized regime. In their work the important parameter is the energy.

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However, for genuine random models, there is no rigorous proof of the existence of a transition or even of the appearance of spectral components other than pure point, so far.

This is a quite strange situation: the unperturbed problem exhibits extended states and purely a.c. spectrum but for the perturbed one can prove the opposite spectral behavior only. overview

Sparse Potentials

As a typical example let us consider the following model in $L^2(\mathbb{R}^d)$,

$$H(\omega) = -\Delta + V_\omega, \text{ where } V_\omega(x) = \sum_{k\in\mathbb{Z}^m} \xi_k(\omega) f(x-k),$$

$f \leq 0$ is a compactly supported single site potential and the ξ_k are independent Bernoulli variables with $p_k := \mathbb{P}\{\xi_k = 1\}$.

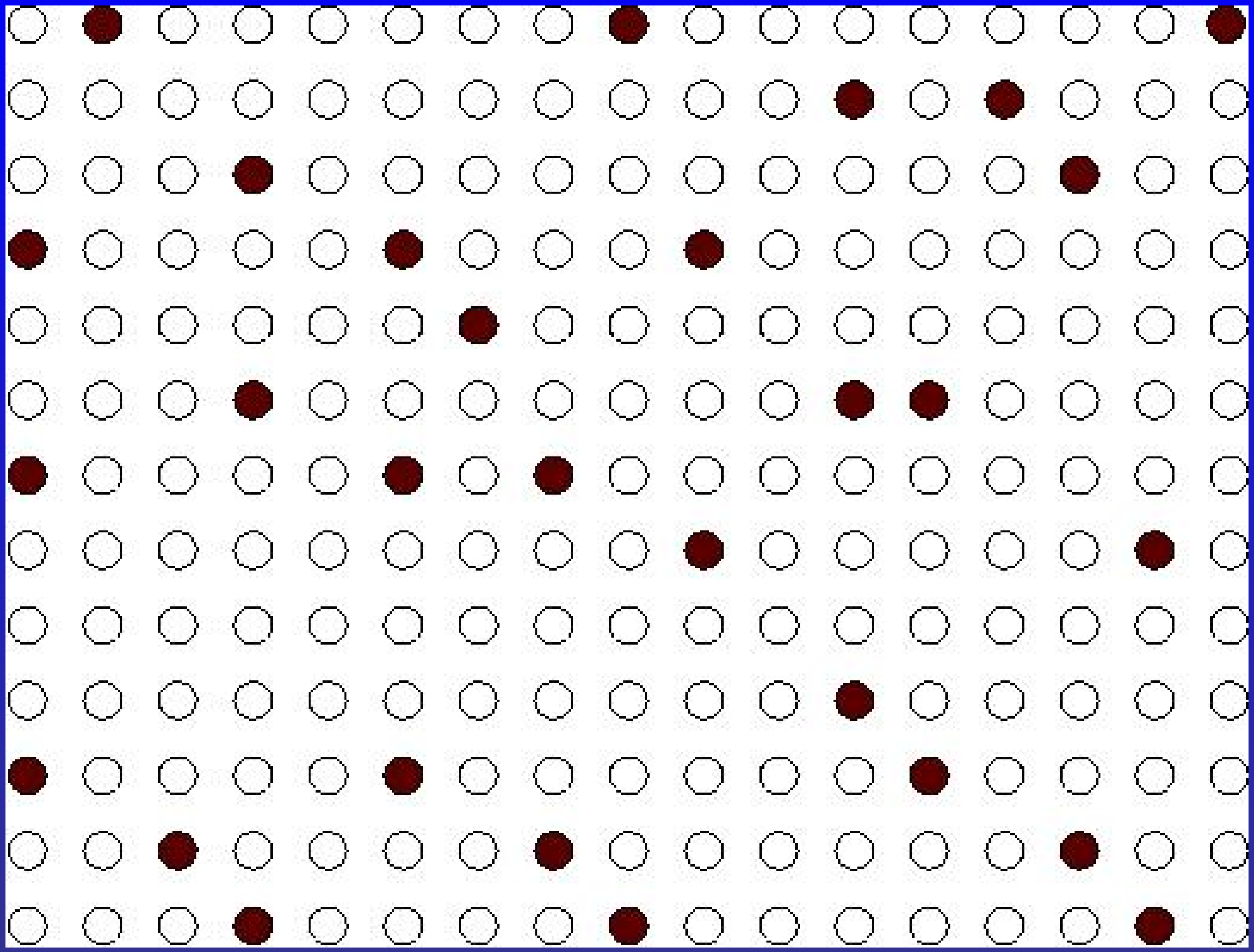

Figure 2: A typical random sparse potential: the red holes are at lattice points k where $\xi_k = 1$

To understand the appearance of a metallic regime, we recall the following facts from scattering theory:

Cook's criterion

We write $-\Delta = H_0$ so that the operators we are interested in can be written as $H = H_0 + V$. By $\sigma_{ac}(H)$ we denote the absolutely continuous spectrum, related to delocalized states.

Theorem 1. *(Cooks criterion)*
If for some $T_0 > 0$ and all ϕ in a dense set

$$\int_{T_0}^{\infty} \|Ve^{-itH_0}\phi\| dt < \infty \qquad (*)$$

then $\Omega_- := lim_{t\to\infty} e^{itH} e^{-itH_0}$ exists and, consequently, $[0, \infty) \subset \sigma_{ac}(H)$, i.e., there are scattering states for H and any nonnegative energy.

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However, the following nice result holds; see Hundertmark and Kirsch [10] who also provided the absolutely correct name:

Theorem 2. *(Almost surely free lunch theorem)*
Assume that

$$W(x) := \left(\mathbb{E}(V_\omega(x)^2)\right)^{\frac{1}{2}} \stackrel{!}{\leq} C(1+|x|)^{-(1+\epsilon)}.$$

Then V_ω satisfies Cook's criterion for a.e. ω.

Proof.

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If we fix $d \geq 3$ and $\frac{d}{2} + \frac{1}{2} < \alpha < d$ and $p_k \sim k^{-\alpha}$ we can moreover control the essential spectrum below 0. We can summarize this in the following picture: overview

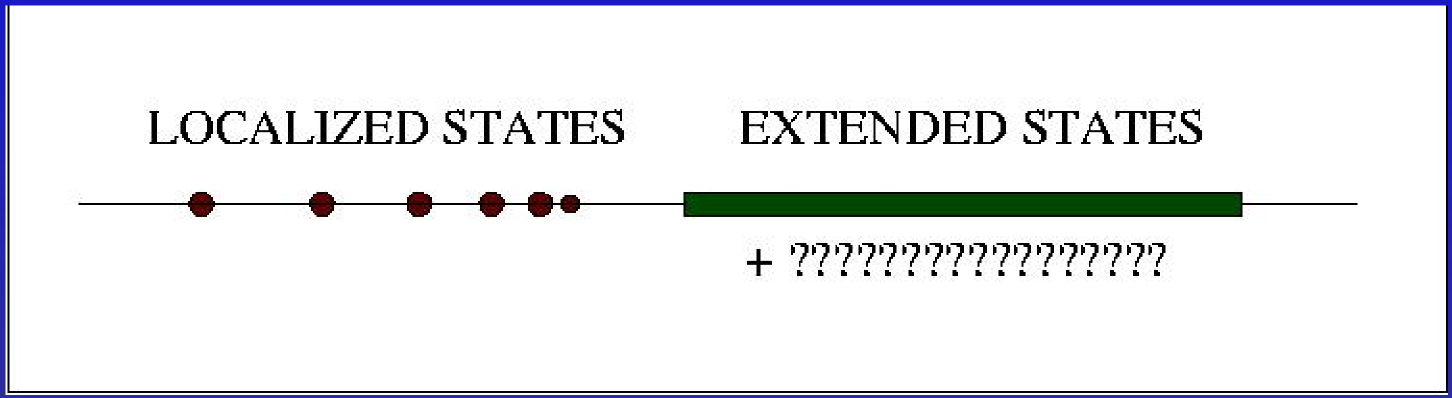

Figure 3: Conclusion and open problems for the sparse model

Discrete and continuum surface models

Consider the following self-adjoint random operator in $L^2(\mathbb{R}^d)$ or $\ell^2(\mathbb{Z}^d)$, $\mathbb{R}^d = \mathbb{R}^m \times \mathbb{R}^{d-m}$:

$$H(\omega) = -\Delta + V_\omega, \text{ where } V_\omega(x) = \sum_{k \in \mathbb{Z}^m} q_k(\omega) f(x - (k, 0)),$$

the q_k are i.i.d. random variables and $f \geq 0$ is a single site potential that satisfy certain technical assumptions. This leads to the following geometry characterizing random surface models. Sometimes the upper half plane is considered only.

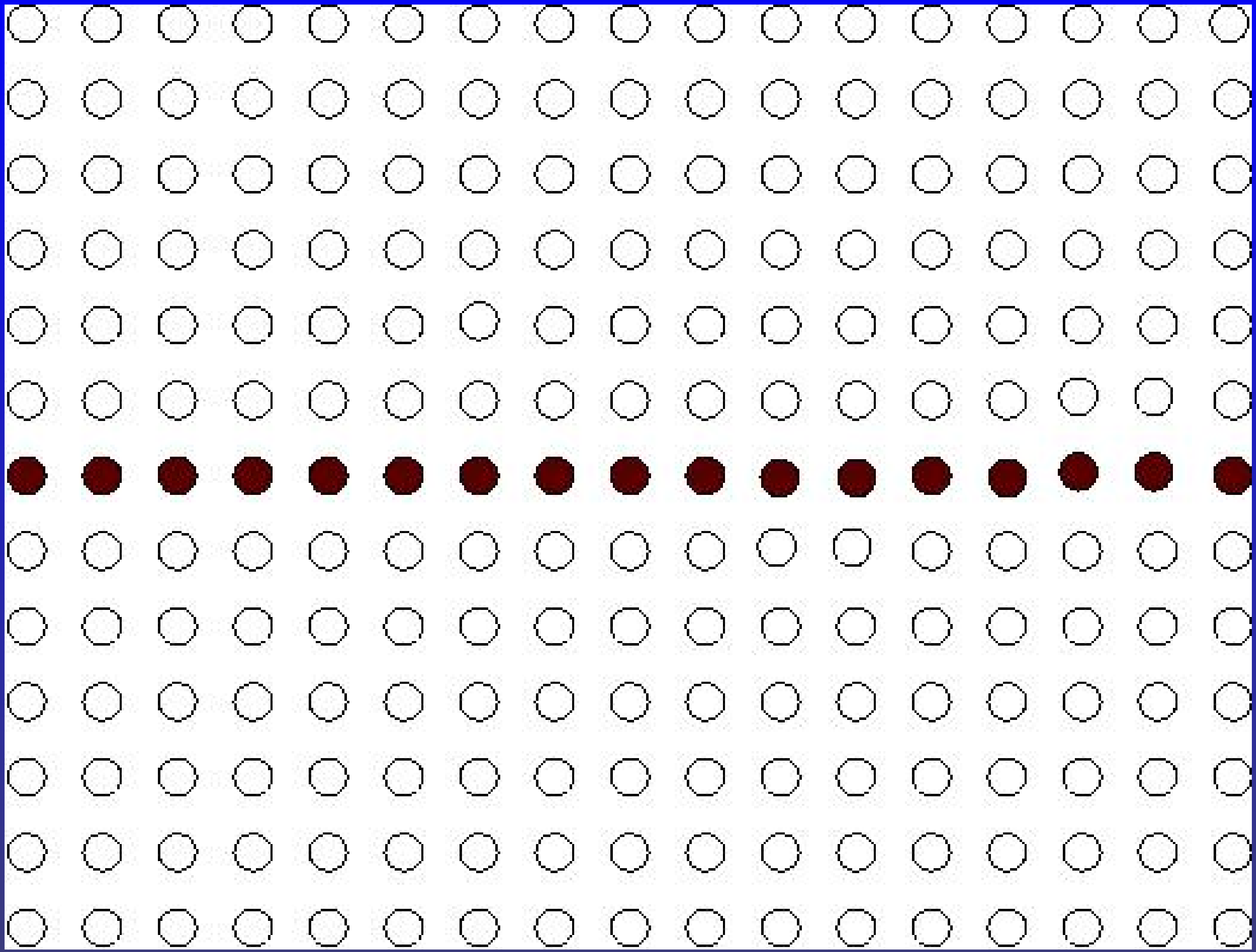

Figure 4: The structure of a surface potential

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The moral of the story is the appearance of a metal insulator transition at the edges of the unperturbed operator.

The continuum model

We now concentrate on the continuum case, where we have the following picture:

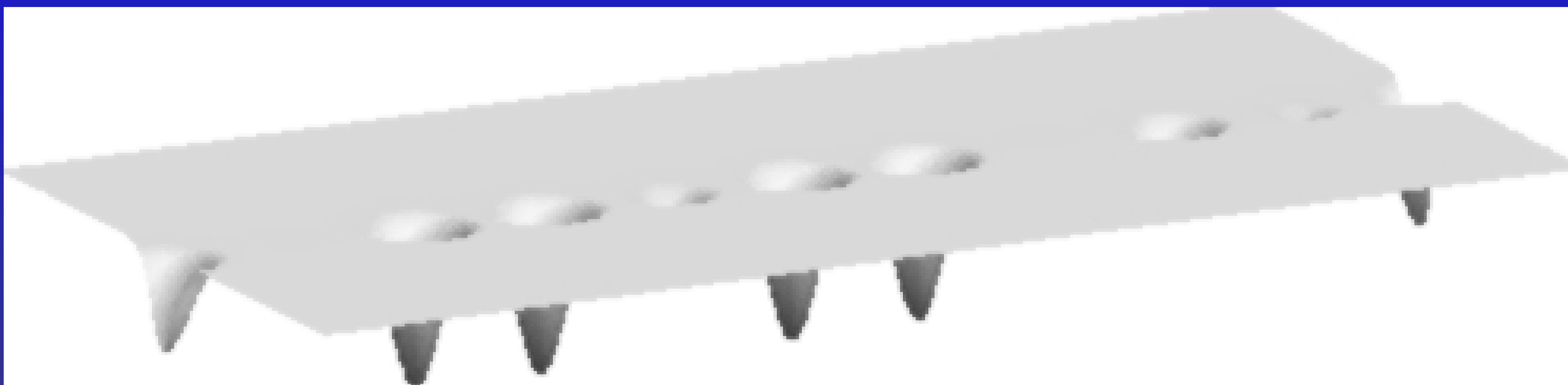

Figure 5: A typical realization of a continuum random surface potential

The spectrum

It is not hard to see that

$$\sigma(H(\omega)) = [E_0, \infty) \text{ where } E_0 = \inf \sigma(-\Delta + q_{\mathsf{min}} \cdot f^{\mathsf{per}}),$$

and

$$f^{\mathsf{per}} = \sum_{k \in \mathbb{Z}^m} f(x - (k, 0))$$

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For nonnegative energies one expects extended states.

Extended states

To stress the existence of a metallic phase let us cite Theorem 4.3 of [10]:

Theorem 3. *Let* $H(\omega)$ *satisfy the hidden assumptions. Then we have, for every* $\omega \in \Omega$*:* $[0, \infty) \subset \sigma_{\mathsf{ac}}(H(\omega))$.

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Theorem 3. *Let* $H(\omega)$ *satisfy the hidden assumptions. Then we have, for every* $\omega \in \Omega$*:* $[0, \infty) \subset \sigma_{\mathrm{ac}}(H(\omega))$.

The idea of the *Proof* is that a wave packet with velocity pointing away from the surface will escape the influence of the surface potential and is asymptotically free. The rigorous implementation of this idea uses Enss' technique from scattering theory.

Localized states

What follows is the main result of the joint paper with A. Boutet de Monvel:

Theorem 4. *Let $H(\omega)$ be as in the above assumptions with $\tau > d/2$.*

(a) *There exists an $\varepsilon > 0$ such that in $[E_0, E_0 + \varepsilon]$ the spectrum of $H(\omega)$ is pure point for almost every $\omega \in \Omega$, with exponentially decaying eigenfunctions.*

(b) *Assume that $p < 2(2\tau - m)$. Then there exists an $\varepsilon > 0$ such that in $[E_0, E_0 + \varepsilon] = I$ we have strong dynamical localization in the sense that for every compact set $K \subset \mathbb{R}^d$:*

$$\mathbb{E}\{\sup_{t>0} \||X|^p e^{-itH(\omega)} P_I(H(\omega))\chi_K\|\} < \infty$$

A consequence is pure point spectrum in the interval $[E_0, E_0 + \varepsilon] = I$. Together with the result on extended states we get the following picture that leaves open some important questions.

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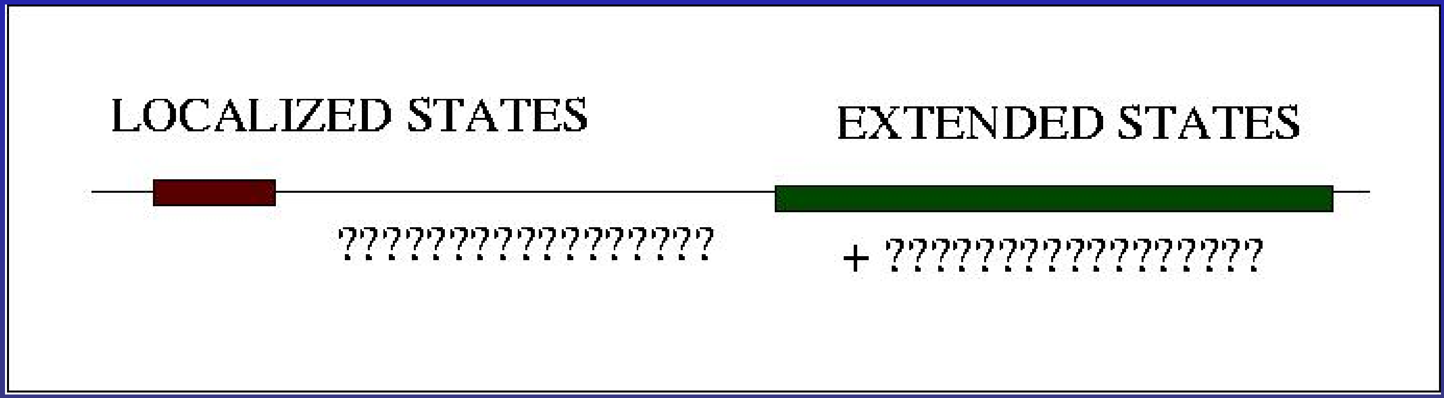

Figure 6: Conclusion and open problems for the continuum surface model

Idea of the proof of localization

The multiscale scheme is an inductive procedure to prove exponential decay estimates for the (box) resolvent. The induction is from one length scale ℓ to the next (much higher, typically ℓ^α).

The starting point (initial length scale estimates) comes from absence of spectrum and a small divisor condition (Wegner estimate) is needed in addition.

The proof of both these latter properties needs an additional idea in the nonstationary case.

Assumptions

(1) $0 < m < d$ and points in $\mathbb{R}^d = \mathbb{R}^m \times \mathbb{R}^{d-m}$ are written as pairs, if convenient;

(2) The single site potential $f \geq 0$, $f \in L^p(\mathbb{R}^d)$ where $p \geq 2$ if $d \leq 3$ and $p > d/2$ if $d > 3$, and $f \geq \sigma > 0$ on some open set $U \neq \emptyset$ for some $\sigma > 0$.

(3) The q_k are i.i.d. random variables distributed with respect to a probability measure μ on $\mathbb{R}$, such that $\operatorname{supp} \mu = [q_{\mathsf{min}}, 0]$ with $q_{\mathsf{min}} < 0$.

We will sometimes need further assumptions on the single site distribution μ:

(4) μ is *Hölder continuous*, i.e. there are constants $C, \alpha > 0$ such that

$$\mu[a, b] \leq C(b-a)^{\alpha} \text{ for } q_{\mathsf{min}} \leq a \leq b \leq 0.$$

(5) *Disorder assumption*: there exist $C, \tau > 0$ such that

$$\mu[q_{\mathsf{min}}, q_{\mathsf{min}} + \varepsilon] \leq C \cdot \varepsilon^{\tau} \text{ for } \varepsilon > 0.$$

back to the model
back to localization
back to extended states

Literature on surface potentials

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