Between order and disorder: a review featuring the integrated density of states

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A very basic yet interesting quantity: the integrated density of states, IDS.

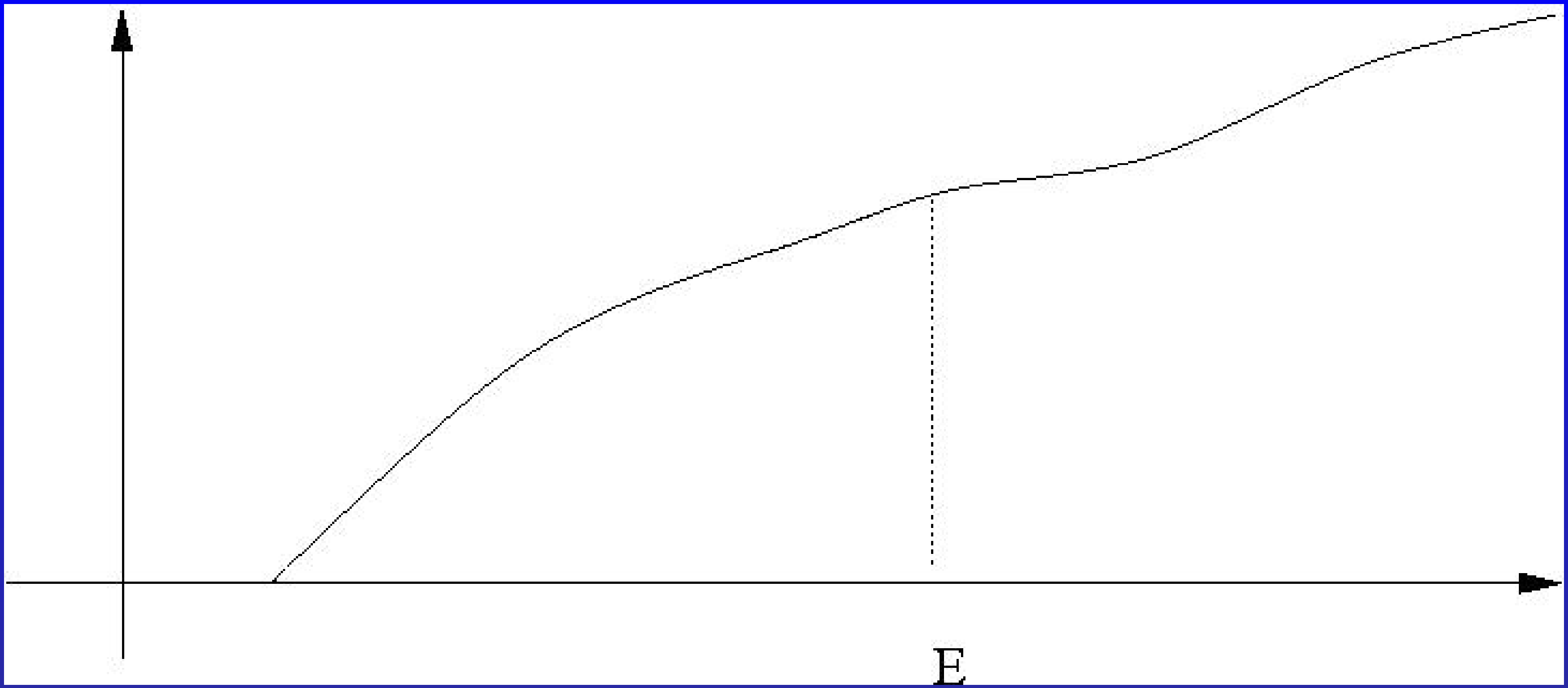

Figure 1: The IDS

$$N(E) := \frac{\#\text{states } \leq E \text{ of } H_\omega}{\text{unit volume}}$$

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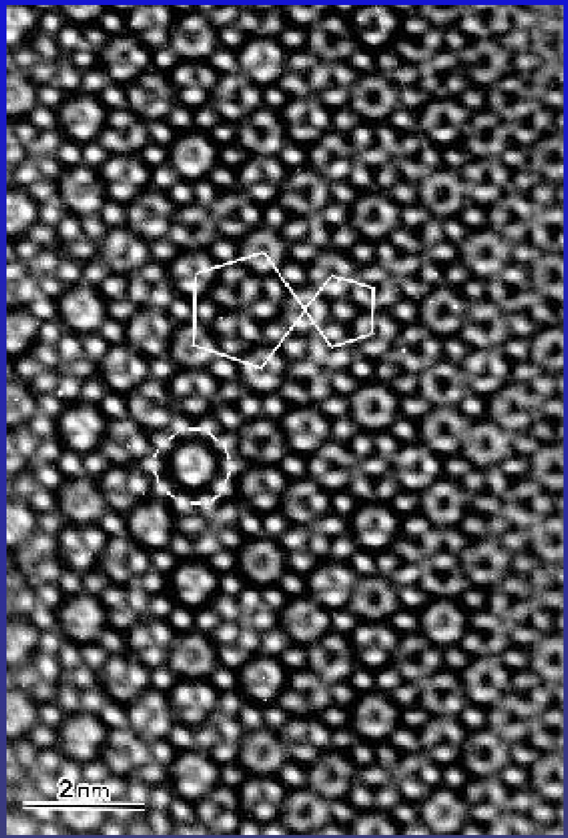
- Asymptotics near spectral edges
  $\longrightarrow$ Lifshitz tails

2. Quasicrystals, tilings and Delone sets

Discovery of quasicrystals was an important trigger for the analysis of aperiodic order.



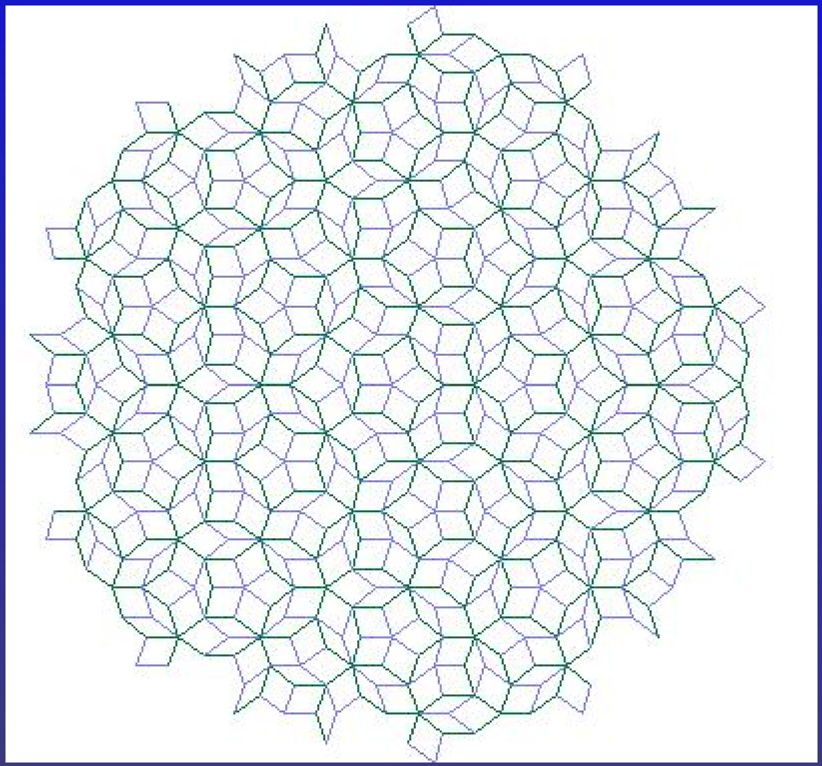


Figure 2: Original und Fälschung

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$$N(E) = \int_\Omega tr[M_f \chi_{(-\infty,E]}(H_\omega)] d\mu(\omega)$$

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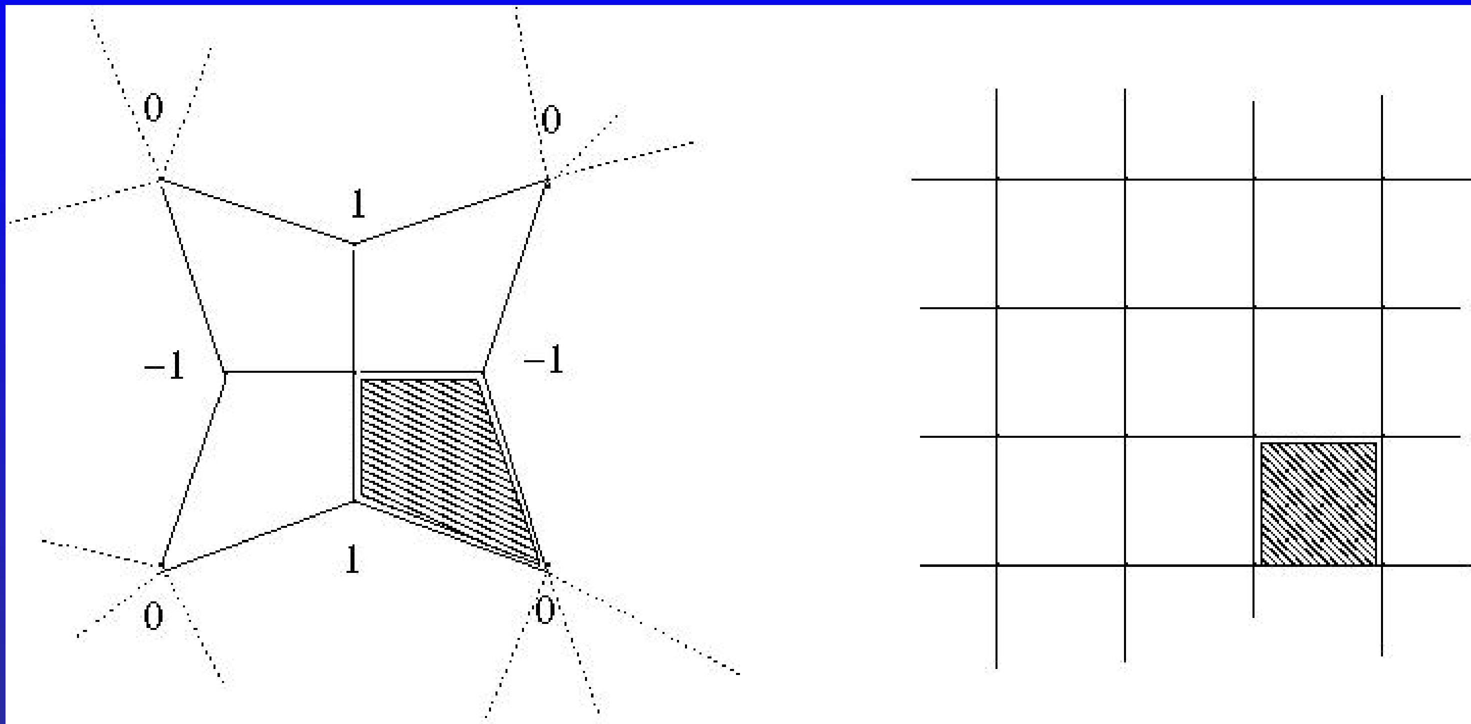
$N(E)$ is discontinuous at E_0
$\Longrightarrow \exists$ a compactly supported eigenfunction, **scar**, for H_ω and E_0.

Curvature and scars



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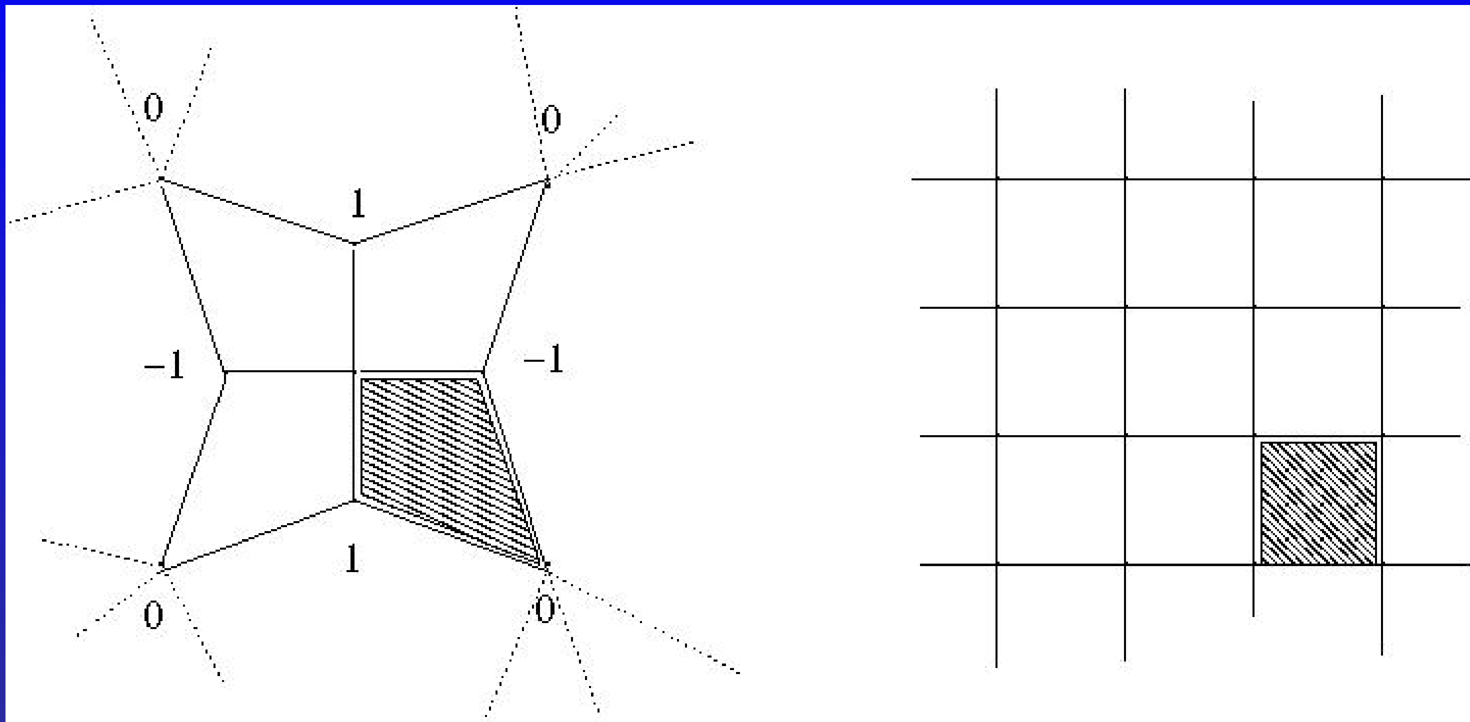

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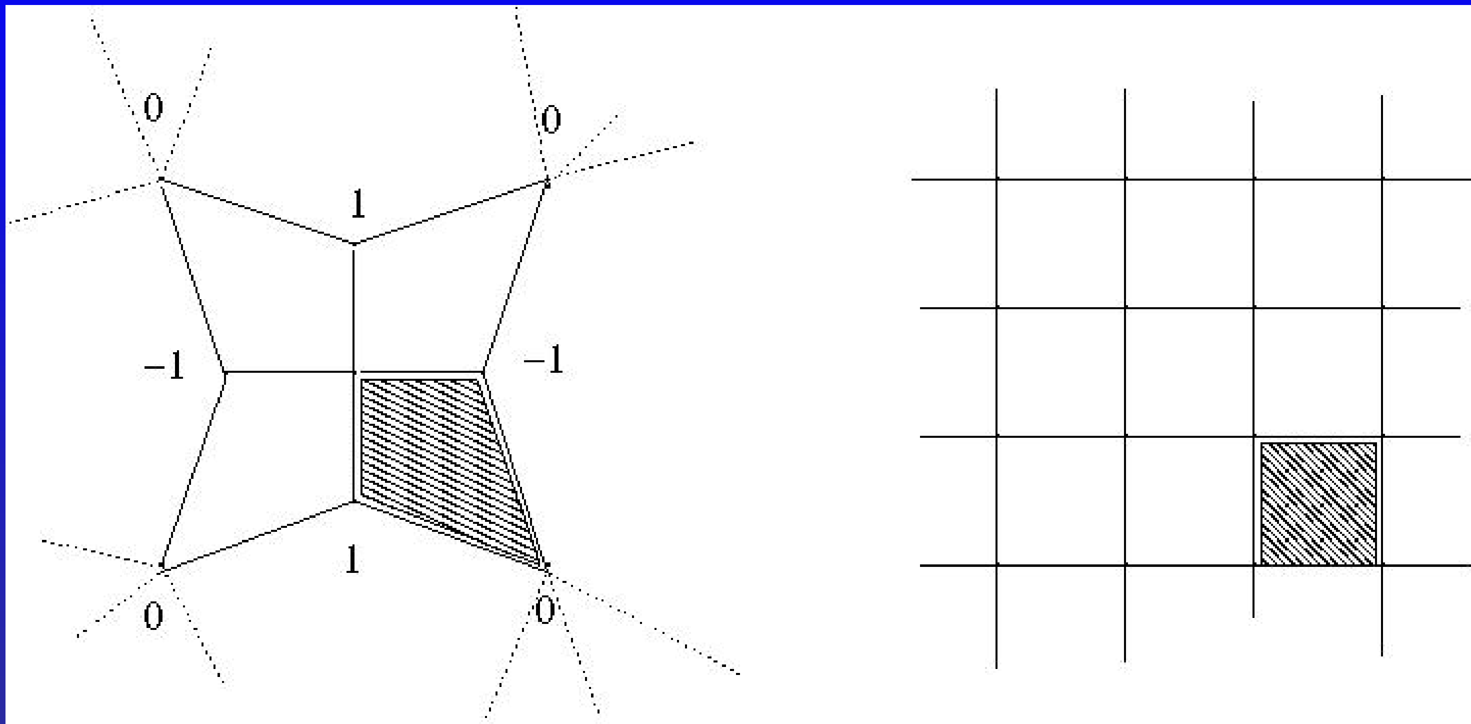

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Klassert, Lenz, Peyerimhoff, S.:

$\kappa(v,f) \leq 0$ for all $(v,f) \Longrightarrow$ Exist **no** scars

3. Percolation graphs

Starting from the *vertex set* $\mathbb{Z}^d$ and *edge set* $\mathbb{E}^d$ we let $\Omega = \{0,1\}^{\mathbb{E}^d}$ and G_ω the graph with edges specified by $\omega(e) = 1$ (bond percolation).

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The IDS exists and satisfies

$$N(E) = \int_\Omega \mathbb{P}(d\omega)\ \langle \delta_0, \chi_{(-\infty,E]}(H_\omega)\delta_0 \rangle$$

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Lifshitz tails for N (Kirsch, Müller for $p < p_c$, Müller, S. for $p \geq p_c$.)

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No Wegner estimate; but localization holds trivially for $p < p_c$.

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An *alloy type model* is a random Schrödinger operator $H_\omega = H_0 + V_\omega$, where $H_0 = H_A + V_{per}$ with a periodic potential V_{per}.
The random part of the potential has the form $V_\omega(x) = \sum_{k \in \mathbb{Z}^d} \omega_k \, u(x-k)$.
The *coupling constants* $\omega_k, k \in \mathbb{Z}^d$, are a sequence of bounded random variables, which are independent and identically distributed with distribution μ. The expectation of the product measure $\bigotimes_{k \in \mathbb{Z}^d} \mu$ is denoted by $\mathbb{E}$. The single *site potential* $u \not\equiv 0$ is of compact support.

Continuity properties

Denote for $\varepsilon > 0$

$$s(\mu, \varepsilon) = \sup\{\mu([E - \varepsilon, E + \varepsilon]) \mid E \in \mathbb{R}\}.$$

With this definition, we have (Hundertmark,Killip, Nakamura, S., Veselic)

$$\mathbb{E}\{tr[\chi_{[E-\varepsilon,E+\varepsilon]}(H_\omega|_{C_\ell})]\} \leq C_W \; s(\mu, \varepsilon) \, (\log \tfrac{1}{\varepsilon})^d \; |C_\ell|$$

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Not yet published Breakthrough: Bourgain and Kenig show

localization in the 2D Bernoulli-Anderson model.

Asymptotics

Recent results on precise Lifshitz tails by Gärtner and König (strongly related to intermittency) and Metzger (PhD thesis).

Nonstationary models

A lot of interest due to metal-insulator transition: Jaksic, Krishna, Last, Molchanov, Pastur (discrete) and Boutet de Monvel, Böcker, Hundertmark, Kirsch, S., Stolz (continuum).

Conclusion

Different levels of disorder lead to different phenomena (dis/continuity, asymptotics) while certain features are quite universal (existence).