Generic singular continuity for measures and Delone Hamiltonians

Peter Stollmann

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Delone sets and Hamiltonians

Here we apply our abstract analysis to Delone Hamiltonians: these constitute models for geometric disorder. We show that these operators exhibit purely singular continuous spectrum generically.

This is joint work with DANIEL LENZ, TU Chemnitz.

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Figure 1: The potential of $H(\omega)$ for one ω.

Aim: under some mild assumptions there exists a dense G_δ-set of ω's for which $H(\omega)$ exhibits a purely singular continuous spectral component.

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Then, their intersection

$$\{x \in X |\, U \subset \sigma(H(x)), \sigma_{ac}(H(x)) \cap U = \emptyset, \sigma_{pp}(H(x)) \cap U = \emptyset\}$$

is a dense G_δ*-set in* X.

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The *proof* is evident from the fact that $S \to \mathcal{M}_+(S), x \mapsto \delta_x$ is continuous.

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Proof. The densities considered in $\mathcal{K}$ form a closed subset of the unit ball of $L^2(K,\lambda)$.
Since the latter is weakly compact and the mapping $L^2(K,\lambda)_+ \longrightarrow \mathcal{M}_+(S), f \mapsto f\cdot\lambda$ is w-w* -continuous we get the desired compactness. □

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Corollary 6. *Let* U *be an open subset of* $\mathbb{R}^d$*. Then the singular continuous measures* $\mathcal{M}_{sc}(U)$ *form a dense* G_δ *in the space* $\mathcal{M}_+(U)$ *of Borel measures.*

Corollary 6. *Let U be an open subset of $\mathbb{R}^d$. Then the singular continuous measures $\mathcal{M}_{sc}(U)$ form a dense G_δ in the space $\mathcal{M}_+(U)$ of Borel measures.*

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By $\mathbb{D}_{r,R}(\mathbb{R}^d) = \mathbb{D}_{r,R}$ we denote the set of all (r, R)-sets. We say that $\omega \subset \mathbb{R}^d$ is a *Delone set*, if it is an (r, R)-set for some $0 < r \leq R$ so that $\mathbb{D}(\mathbb{R}^d) = \mathbb{D} = \bigcup_{0 < r \leq R} \mathbb{D}_{r,R}(\mathbb{R}^d)$ is the set of all Delone sets.

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where $\omega \in \mathbb{D}$, $v \geq 0$ is bounded, measurable and compactly supported. If $\rho \in \mathbb{D}_{r,R}$ is $crystallographic$, i.e.

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Theorem 7. *Let* $r, R > 0$ *with* $2r < R$ *and* $v \neq 0$. *Then there exists an open* $\emptyset \neq U \subset \mathbb{R}$ *and a dense* G_δ*-set* $\Omega_{sc} \subset \mathbb{D}_{r,R}$ *such that for every* $\omega \in \Omega_{sc}$ *the spectrum of* $H(\omega)$ *contains* U *and is purely singular continuous in* U.

Proof. $\mathbb{D}_{r,R} =: X$, use the “Wonderland” theorem.

Consider

$$H(\omega) := -\Delta + \sum_{x\in\omega} v(\cdot - x) \text{ in } \mathbb{R}^d,$$

where $\omega \in \mathbb{D}$, $v \geq 0$ is bounded, measurable and compactly supported. If $\rho \in \mathbb{D}_{r,R}$ is *crystallographic*, i.e.

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is a lattice, then $H(\rho)$ is periodic.

Theorem 7. *Let $r, R > 0$ with $2r < R$ and $v \neq 0$. Then there exists an open $\emptyset \neq U \subset \mathbb{R}$ and a dense G_δ-set $\Omega_{sc} \subset \mathbb{D}_{r,R}$ such that for every $\omega \in \Omega_{sc}$ the spectrum of $H(\omega)$ contains U and is purely singular continuous in U.*

Proof. $\mathbb{D}_{r,R} =: X$, use the "Wonderland" theorem.

Since $2r < R$, there exist crystallographic $\gamma, \tilde{\gamma}$ such that $U := \sigma(H(\gamma))^\circ \setminus \sigma(H(\tilde{\gamma})) \neq \emptyset$. picture

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To this end consider $\omega_n \in \mathbb{D}_{r,R}$ s.t. $\omega_n \cap [-n,n]^d = \omega \cap [-n,n]^d$ and $\omega_n \cap ([-n-R, n+R]^d)^c = \tilde{\gamma} \cap ([-n-R, n+R]^d)^c$, picture

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Similarly, the other sets are shown to be dense and the Wonderland theorem implies the assertion. $\square$

Conclusion

- $\mathcal{M}_c(S)$ and $\mathcal{M}_s(S)$ are G_δ-sets, for polish S.
- This implies the Wonderland theorem and the fact that generic measures are singular continuous in “nice spaces”.
- A particular example is given by “geometric disorder” ($=$ Delone Hamiltonians).

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Reminder:

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$$\Longrightarrow$$

$$\rho_\xi^{A_n} \to \rho_\xi^{A} \text{ vaguely, where } \langle \rho_\xi^A, \varphi \rangle := (\varphi(A)\xi|\xi).$$

back

Delone sets

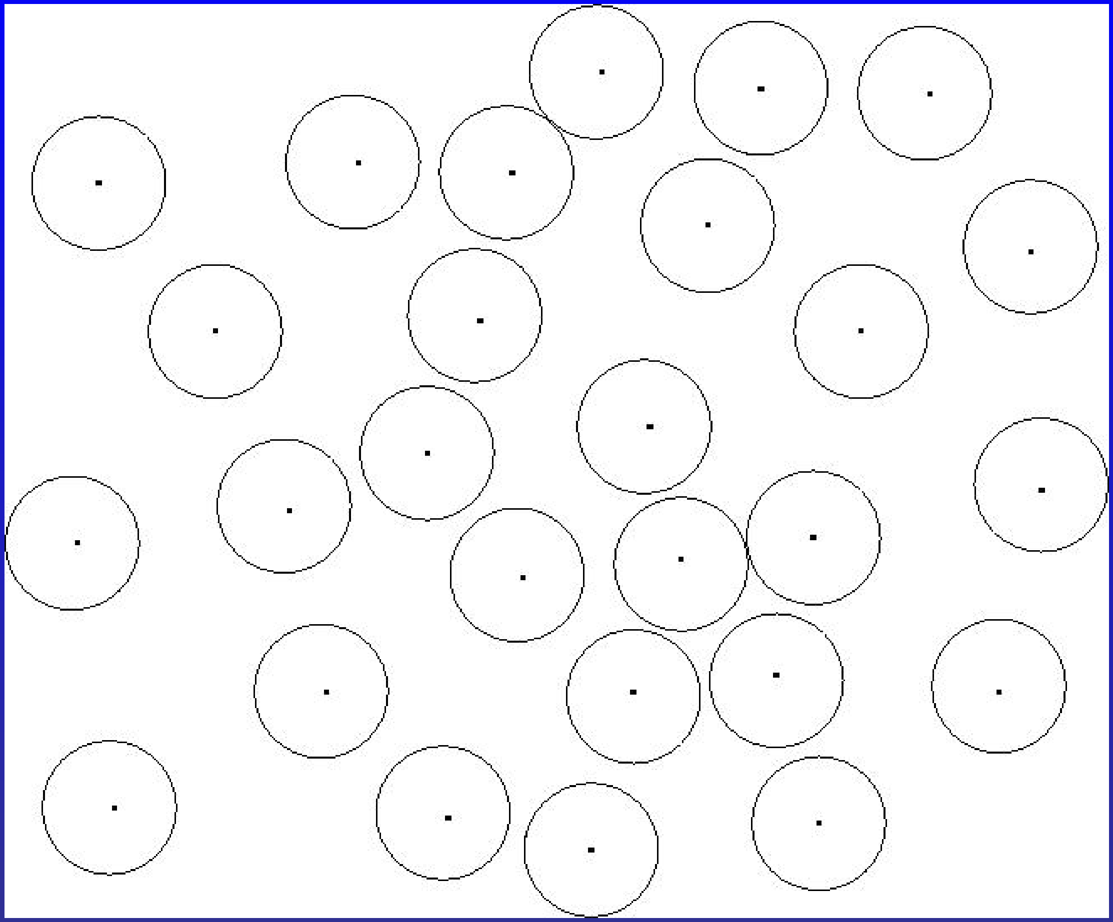

Figure 2: A Delone set ω and r.

back

γ and $\tilde{\gamma}$

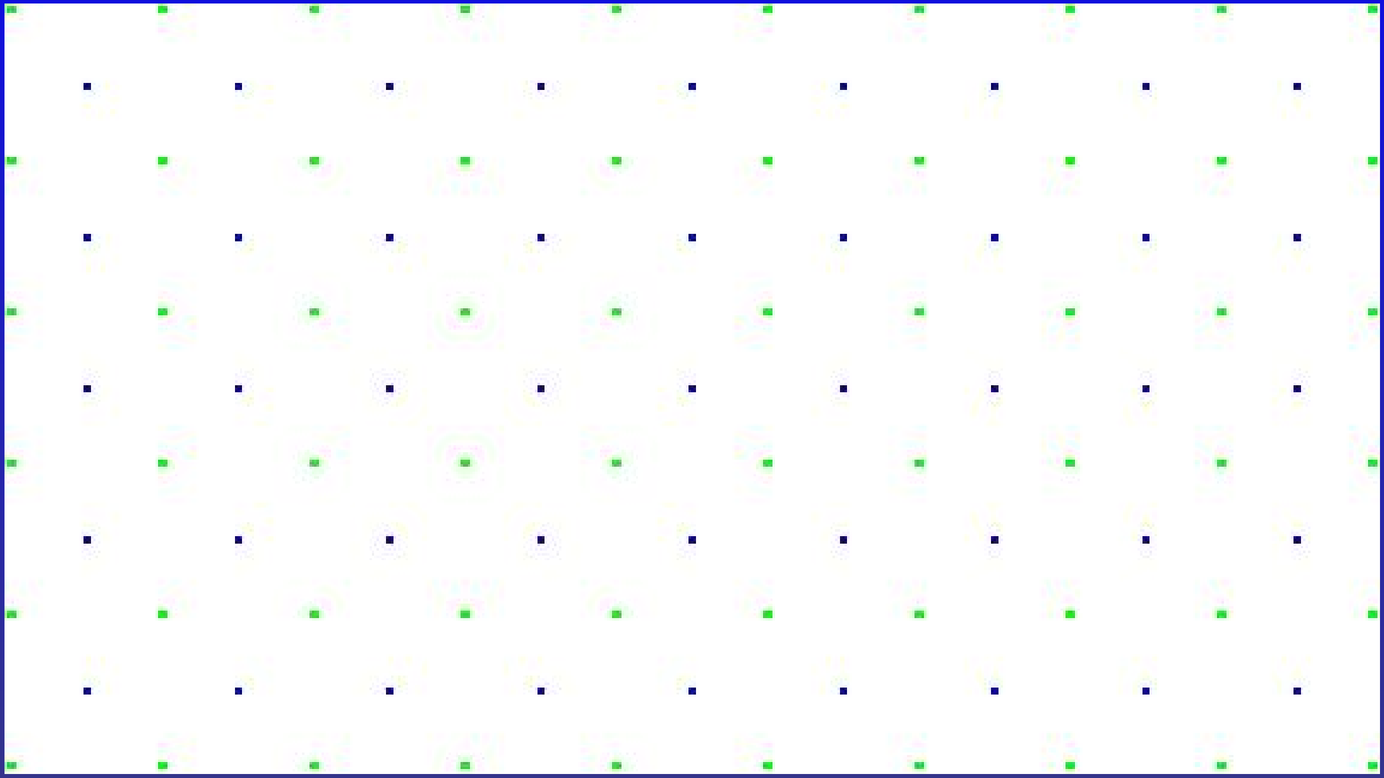

Figure 3: crystallographic γ (green) and $\tilde{\gamma}$ (green+blue) back

ω_n

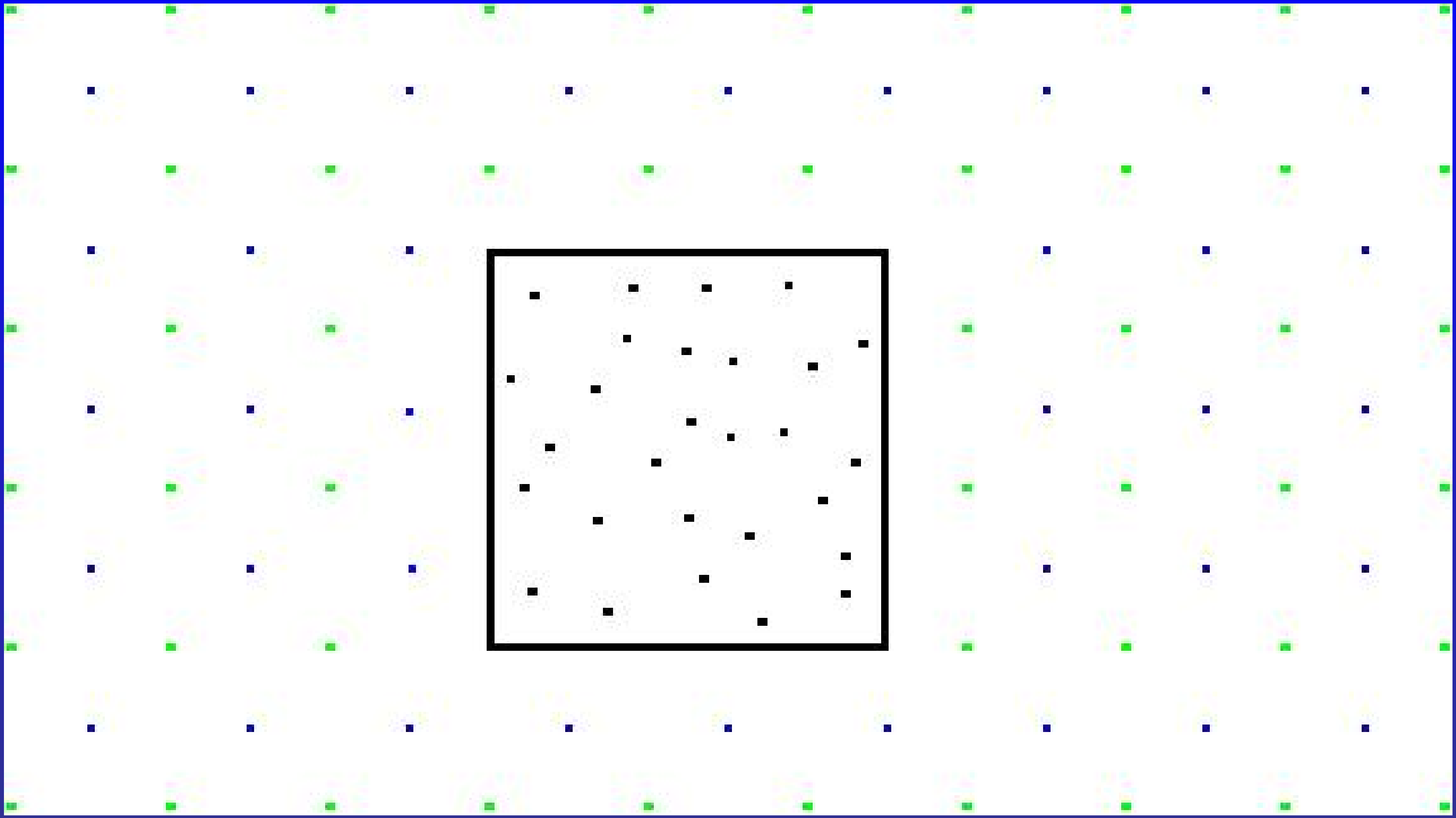

Figure 4: ω_n back