# The Caccioppoli inequality for general Schrödinger operators

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# Overview

1. Caccioppoli??... and why should one care?
2. Dirichlet forms: a short course feat. the energy measure and the intrinsic metric.
3. Examples
4. The proof of the Caccioppoli inequality
5. More results

A remix of results obtained in collaboration with Anne Boutet de Monvel, Daniel Lenz, Ivan Veselić

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# Dirichlet forms

$X$ a second countable locally compact space
$m$ a Borel measure
$\mathcal{E}$ a strongly local Dirichlet form
with energy measure $\Gamma(u, v)$, so that

$$\mathcal{E}(u, v) = \int_X d\Gamma(u, v)$$

$V = \mu_+ - \mu_-$ where $\mu_-$ is form small w.r.t. $\mathcal{E}$

Theorem
*If $u$ is a weak solution of $(H - V)u = \lambda u$ then:*

$$\int_E d\Gamma(u, u) \leq \frac{C}{r^2} \int_{B_r(E)} |u|^2 dm$$

*holds for any closed $E \subset X$ and any $r > 0$.*

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# Consequences of locality

Leibniz rules: $d\Gamma(u \cdot v, w) = ud\Gamma(v, w) + vd\Gamma(u, w)$.
Chain rule: $d\Gamma(\zeta \circ u, w) = \zeta' \circ ud\Gamma(u, w)$.
Cauchy - Schwarz:

$$\int_X |fg| d|\Gamma(u, v)| \leq \frac{1}{2} \int_X |f|^2 d\Gamma(u) + \frac{1}{2} \int_X |g|^2 d\Gamma(v).$$

For a weak solution $(H - V)u = \lambda u$, we have

$$\int \eta^2 d\Gamma(u) = \int (\lambda - V)(|\eta u|^2) - 2 \int \eta\, u\, d\Gamma(\eta, u).$$

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# More results

- Combes - Thomas estimate for $H + V$ in terms of the intrinsic metric. Eigenfunction expansion for $H + V$ (Boutet de Monvel, S. '03).
- Sch'nol's theorem for $H + V$ (Boutet de Monvel, Lenz, S. '08).
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