# The complex Laplacian and its heat semigroup

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# Overview

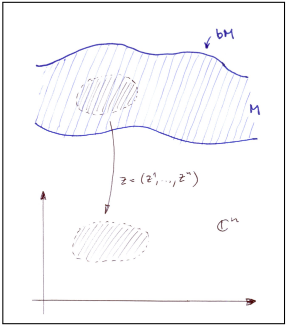
- The $\bar\partial$ Neumann operator - an introduction.
- The definition of the complex Laplacian $\Box$.
- The basic estimate.
- A Nash-type inequality.
- More results.

Based on


- J.J. Perez and P.S.: Heat kernel estimates for the $\bar\partial$-Neumann problem on $G$-manifolds, Manuscripta Math., to appear
- J.J. Perez and P.S.: Essential self-adjointness, generalized eigenforms, and spectra for the $\bar\partial$-Neumann problem on $G$-manifolds, *J. Funct. Anal.* 261 (2011), pp. 2717-2740 and available at http://www.mat.univie.ac.at/~esiprpr/esi2301.pdf

# The set-up

Area: Several complex variables (SCV) and complex geometry and PDE. The $\bar\partial$ Neumann Laplacian $\Box$ is the analogue for complex manifolds of the Hodge Laplacian of Riemannian manifolds. It acts on spaces of forms.




$M$ a complex manifold with boundary $bM$, local coordinates $z = (z^1, ..., z^n)$.
Classical: $\overline{M} = M \cup bM$ compact.
Our results: $\overline{M}$ invariant under a free group action with compact quotient.

# $(p, q)$-forms

The complex structure comes with

$$\frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right), j = 1, \dots, n$$

$$dz^j = dx^j + idy^j, d\bar{z}^j = dx^j - idy^j, j = 1, \dots, n.$$

We now introduce $(p, q)$-forms:

$$u = \sum_{|I|=p, |J|=q} u_{I,J}\ dz^I \wedge d\bar{z}^J \in C^\infty(M, \Lambda^{p,q}),$$

where $dz^I = dz^{i_1} \wedge \cdots \wedge dz^{i_p}$, $dz^J = d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$, $I = (i_1, \dots, i_p)$, $J = (j_1, \dots, j_q)$, $i_1 < \cdots < i_p$, $j_1 < \cdots < j_q$, and the $u_{I,J}$ are smooth functions in local coordinates.

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# The antiholomorphic exterior derivative $\bar\partial$

For

$$u = \sum_{|I|=p,|J|=q} u_{I,J}\ dz^I \wedge d\bar z^J \in C^\infty(M, \Lambda^{p,q})$$

let

$$\bar\partial u = \sum_{|I|=p,|J|=q} \sum_{k=1}^{n} \frac{\partial u_{I,J}}{\partial \bar z^k}\ d\bar z^k \wedge dz^I \wedge d\bar z^J.$$

With respect to a smooth measure on $M$ and a smoothly varying Hermitian structure in the fibers of the tangent bundle, define the spaces $L^2(M, \Lambda^{p,q})$. Let us extend the above $\bar\partial$ to the corresponding maximal operator in $L^2$ (and still call it $\bar\partial$) and let $\bar\partial^*$ be its adjoint operator (the differential forms in the domain of $\bar\partial^*$ will have to satisfy certain boundary conditions) ... later.

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# The operator $\square$.

The central operator is now

$$\square = \bar\partial^* \bar\partial + \bar\partial \bar\partial^*,$$

defined as the form sum. Standard reference

Folland, G.B., Kohn, J.J.: The Neumann Problem for the Cauchy-Riemann Complex, *Ann. Math. Studies*, **75** Princeton University Press, Princeton, N.J. 1972

Classical work: F. Hartogs, E.E. Levi, K. Oka, K. Stein, H. Grauert ...(SCV, sheaf theory).

# The operator $\Box$.

J.J. Kohn, L. Hörmander, L. Nirenberg, C. Fefferman, E.M. Stein ... (PDE).

Gromov, Henkin and Shubin (noncompact).
We rely additionally on work of E. Straube, M. Engliš, J.J. Perez.

# The form domain of $\Box$.

*"It seems that K. Friedrichs was the first to realize that the shortest distance to the heart of a symmetric, non-negative definite operator is through its quadratic form"*
(D. Stroock).

In our case,

$$\Box = \Box_{p,q} = \bar\partial^*\bar\partial + \bar\partial\bar\partial^*$$

is the selfadjoint operator in $L^2(M, \Lambda^{p,q})$ associated with

$$\begin{aligned} \mathrm{dom}(Q^{p,q}) &:= \mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*), \\ Q^{p,q}(u,u) &:= \langle \bar\partial u, \bar\partial u\rangle_{L^2(M,\Lambda^{p,q+1})} + \langle \bar\partial^* u, \bar\partial^* u\rangle_{L^2(M,\Lambda^{p,q-1})} \end{aligned}$$

The domain of the corresponding operator $\Box_{p,q}$ is

$$\mathrm{dom}(\Box_{p,q}) = \{u \in \mathrm{dom}(Q^{p,q}) \mid \bar\partial u \in \mathrm{dom}(\bar\partial^*), \bar\partial^* u \in \mathrm{dom}(\bar\partial)\}.$$

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# The form domain of $\Box$.

The condition $u \in \mathrm{dom}(\bar\partial^*)$ is determined by an integration by parts; for $u \in C^\infty(M, \Lambda^{p,q})$ it reads

$$\sum_{k=1,\dots,n} u_{I,kK} \frac{\partial \rho}{\partial z^k} = 0 \text{ for all } K.$$

Here $\rho$ is the function defining the boundary.
NOTE: $u \in C^\infty(\overline{M}, \Lambda^{p,q}) \cap \mathrm{dom}(\bar\partial^*)$ and $\varphi \in C^\infty(\overline{M}, \mathbb{C}) \Longrightarrow \varphi u \in \mathrm{dom}(\bar\partial^*)$. By continuity this gives:

## Proposition

*Let $\varphi \in W^{1,\infty}(\overline{M}, \mathbb{R})$ and $u \in \mathrm{dom} Q$. Then $\varphi u \in \mathrm{dom} Q$ and*

$$Q(\varphi u) \lesssim \|\phi\|^2_{W^{1,\infty}} \left[ Q(u) + \|u\|^2_{L^2} \right].$$

This is important for our definition and use of the *intrinsic metric*.

# The basic estimate.

Due to the boundary, the operator is *not elliptic*; we are concerned with a *noncoercive boundary value* problem. The basic estimate shows that $\square$ is subelliptic, provided the boundary satisfies a suitable *convexity condition*. Let $\rho$ be a smooth function defining the boundary, i.e.,

$$M = \{z \mid \rho(z) < 0\}, bM = \{z \mid \rho(z) = 0\}$$

and define the *Levi form* $L_x$ by

$$T_x^{\mathbb{C}}(bM) = \{w \in \mathbb{C}^n \mid \sum_{k=1}^{n} \frac{\partial \rho}{\partial z^k} w^k = 0\},$$

$$L_x(v, w) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z^j \partial \bar{z}^k} v^j \bar{w}^k, v, w \in T_x^{\mathbb{C}}(bM).$$

$M$ is called *strongly pseudoconvex*, if $L_x$ is positive definite for every $x \in bM$.

If $M$ is strongly pseudoconvex, one gets (Kohn, Engliš, Perez):

$$\|u\|_{H^{s+1}(M,\Lambda^{p,q})} \lesssim \|\Box u\|_{H^s(M,\Lambda^{p,q})} + \|u\|_{L^2(M,\Lambda^{p,q})},$$

where we need the invariance of $M$ with compact quotient (or some uniformity). There are variants under weaker convexity conditions, see work of d'Angelo and Catlin. In these cases, one has to replace $s+1$ by $s+\varepsilon$. We use a Corollary:

## Corollary

*Let $M$ be a strongly pseudoconvex $G$-manifold on which $G$ acts freely by holomorphic transformations with compact quotient $\overline{M}/G$. For integer $s > \dim_{\mathbb{C}} M$ and $q > 0$ we have the estimate*

$$\|(\Box + 1)^{-s} u\|_{L^\infty(M,\Lambda^{p,q})} \lesssim \|u\|_{L^2(M,\Lambda^{p,q})}, \qquad (u \in L^2(M,\Lambda^{p,q})).$$

# Examples: Tubes in Matrix Groups.

The procedure is from Heinzner, Huckleberry and Kutzschebauch '95 and DellaSalla and Perez '10: For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, define the three-dimensional Heisenberg group

$$\mathbb{H}_3(\mathbb{K}) = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \mid z_k \in \mathbb{K} \right\}.$$

The function $\varphi : \mathbb{H}_3(\mathbb{C}) \to \mathbb{R}$ given by

$$\varphi(Z) = (\mathfrak{Im}\, z_1)^2 + (\mathfrak{Im}\, z_2)^2 + (\mathfrak{Im}\, z_3 - \mathfrak{Re}\, z_2\, \mathfrak{Im}\, z_1)^2$$

is invariant under right multiplication by matrices in $\mathbb{H}_3(\mathbb{R})$. Then

$$M_\epsilon = \{\varphi < \epsilon\} \subset \mathbb{C}^3$$

is strongly pseudoconvex as long as $\epsilon < 1$.

# Examples: An example with no $L^2$-holomorphic functions.

... is taken from Gromov, Henkin and Shubin '98:

$$G_\epsilon := \{(z_1, z_2) \in \mathbb{C}^2 \mid \mathfrak{Im}\, z_2 > (\mathfrak{Re}\, z_1)^2 + \frac{1}{\epsilon^2}(\mathfrak{Im}\, z_1)^2\}$$

on which

$$G = \left\{ \begin{pmatrix} \lambda^2 & 2i\lambda\xi & t + i\xi^2 \\ 0 & \lambda & \xi^2 \\ 0 & 0 & 1 \end{pmatrix} \mid \xi, t \in \mathbb{R}, \lambda > 0 \right\}.$$

acts viz.

$$(z_1, z_2) \mapsto (\lambda z_1 + \xi, \lambda^2 z_2 + t + 2i\lambda\xi z_1 + i\xi^2).$$

# A Nash-type inequality.

Here comes one of our results

## Theorem

*Let $M$ be a strongly pseudoconvex $G$-manifold on which $G$ acts freely by holomorphic transformations with compact quotient $\overline{M}/G$. For integer $s > \dim_{\mathbb{C}} M$*

$$\|u\|^{2+\frac{1}{s}}_{L^2(M,\Lambda^{p,q})} \lesssim Q(u)\|u\|^{\frac{1}{s}}_{L^1(M,\Lambda^{p,q})}, \quad (u \in \mathrm{dom}(Q^{p,q}) \cap L^1(M, \Lambda^{p,q})).$$

For the *Proof* we first use the above Corollary to estimate: For integer $s > \dim_{\mathbb{C}} M$ and $q > 0$, we have

$$\|P_t u\|_{L^\infty(M,\Lambda^{p,q})} \lesssim \max(1, t^{-s})\|u\|_{L^2(M,\Lambda^{p,q})}, \quad (u \in L^2(M, \Lambda^{p,q})).$$

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# Proof of the Nash inequality.

From the above ultracontractivity and duality we get:

$$t^{-2s}\|u\|_{L^1}^2 \geq \langle P_t u, P_t u\rangle_{L^2} = \|P_t u\|_{L^2}^2.$$

We use the fundamental theorem of calculus and

$$\frac{d}{dt}\|P_t u\|_{L^2(M,\Lambda^{p,q})}^2 = -2Q(P_t u).$$

to get

$$t^{-2s}\|u\|_{L^1}^2 \geq \|u\|_{L^2}^2 - 2\int_0^t Q(P_s u)ds$$

$$\geq \|u\|_{L^2}^2 - 2tQ(u)$$

by functional calculus. Putting $t = Q(u)^{-\frac{1}{2s+1}}\|u\|_{L^1(M,\Lambda^{p,q})}^{\frac{2}{2s+1}}$ gives the assertion.

# Gaussian decay for the heat semigroup

We can also prove off-diagonal decay of the heat semigroup:

**Theorem**

*Let $M$ be as above. For measurable subsets $A, B$ of $M$ it follows that the heat semigroup satisfies*

$$\| \mathbf{1}_B P_t \mathbf{1}_A \|_{L^2 \to L^2} \leq \exp\left[ -\frac{d(A;B)^2}{4t} \right].$$

For the distance $d$ we use an intrinsic distance that turns out to be the usual Riemannian distance of the underlying real manifold.

## Idea of the proof of Gaussian decay.

Want to estimate

$$\|e^{\delta w} P_t e^{-\delta w}\|_{L^2 \to L^2}$$

for suitable weight function $w$ and $\delta > 0$. Leads to "integrating"

$$Q(e^{-\delta w} u, e^{\delta w} u);$$

... which works by invariance of $\mathrm{dom}(Q)$ under multiplication;

**Lemma**

*For $w \in W^{1,\infty}$ and $u \in \mathrm{dom}(Q)$:*

$$Q(e^{-\delta w} u, e^{\delta w} u) = Q(u, u) + i[\ldots] - \delta^2 (\|\bar{\partial} w \wedge u\|_2^2 + \|\partial w \wedge \star u\|_2^2).$$

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# Idea of the proof of Gaussian decay.

Let $|\bar{\partial} w| \leq 1$, $u \in \mathrm{dom}(Q)$:

$$\begin{aligned} \|e^{\delta w} P_t u\|_2^2 - \|e^{\delta w} u\|_2^2 &= \int_0^t \frac{d}{ds} \|e^{\delta w} P_s u\|_2^2 ds \\ &= -2\,\mathfrak{Re} \int_0^t Q(P_s u, e^{2\delta w} P_s u) ds \\ &\leq 4\delta^2 \int_0^t \|e^{\delta w} P_s u\|_2^2. \end{aligned}$$

By Gronwall's inequality:

$$\|e^{\delta w} P_t u\|_2^2 \leq e^{4\delta^2 t} \|e^{\delta w} u\|_2^2$$

and therefore

$$\|e^{\delta w} P_t e^{-\delta w}\|_{L^2 \to L^2} \leq e^{2\delta^2 t};$$

# Idea of the proof of Gaussian decay.

this implies:

$$\| \mathbf{1}_B P_t \mathbf{1}_A \|_{L^2 \to L^2} \leq e^{2\delta^2 t} e^{-\delta d_{int}(A,B)},$$

where

$$d_{int}(x,y) = \sup\{w(y) - w(x) \mid |\bar{\partial} w| \leq 1\}.$$

This is known as Davies trick. By good luck:

$$d_{int}(x,y) = \sqrt{2} d(x,y).$$

Inserting the right $\delta$ gives the assertion.

# More results.

By a nontrivial cutoff procedure, using ideas of Straube, we get

## Theorem

*Let $M$ be as above. Then $\Box$ is essentially self-adjoint on* $C_c^\infty(\overline{M}, \Lambda^{p,q}) \cap \mathrm{dom}\Box$.

We also have results on expansion in generalized eigenfunctions based on Boutet de Monvel& P.S., '03 and on Shnol's Theorem, based on Boutet de Monvel, Lenz& P.S. '09.