Localization near fluctuation boundaries via the fractional moment method

Peter Stollmann

Prague, 13.9.2006

What's new?

Fluctuation boundaries

The result

Applications

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Outline

This is about joint work with
A. Boutet de Monvel, S. Naboko and G. Stolz
eprint mparc 05-324

- What's new?
- Fluctuation boundaries. This is the key notion.
- The precise results.
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- We present a fractional moment method that works without a covering condition.
- We can treat quite arbitrary geometries, quite arbitrary background ...
- ... as long as the ground state energy is induced by the random perturbation.
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- This notion is classical; it refers to extreme spectral energies that are subtle to changes of the randomness.
- At Fluctuation boundaries one expects Lifshitz tails

In our results we assume $d \leq 3$ and rely upon the following assumptions, which guarantee self-adjointness and lower semi-boundedness of all the Schrödinger operators in question.

(A1) The background potential $V_0 \in L^2_{\text{loc,unif}}(\mathbb{R}^d)$ is real-valued, $H_0 := -\Delta + V_0$.

(A2) The set $\mathcal{I} \subset \mathbb{R}^d$, where the random impurities are located, is uniformly discrete, i.e., $\inf\{|\alpha - \beta| : \alpha \neq \beta \in \mathcal{I}\} =: r_{\mathcal{I}} > 0$.

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General assumptions

(A3) The random couplings η_α, $\alpha \in \mathcal{I}$, are independent random variables supported on $[0, \eta_{\max}]$ for some $\eta_{\max} > 0$ and with absolutely continuous distribution of bounded density ρ_α with a uniform bound $\sup_\alpha \|\rho_\alpha\|_\infty =: M_\rho < \infty$.
The single site potentials U_α, $\alpha \in \mathcal{I}$ satisfy

$$c_U \chi_{\Lambda_{r_U}(\alpha)} \le U_\alpha \le C_U \chi_{\Lambda_{R_U}(\alpha)}$$

for all α with $c_U, C_U, r_U, R_U > 0$ independent of α.

$$V_\omega(x) = \sum_{\alpha \in \mathcal{I}} \eta_\alpha(\omega) U_\alpha(x)$$

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$$H := H(\omega) := H_0 + V_\omega \text{ in } L^2(\mathbb{R}^d).$$

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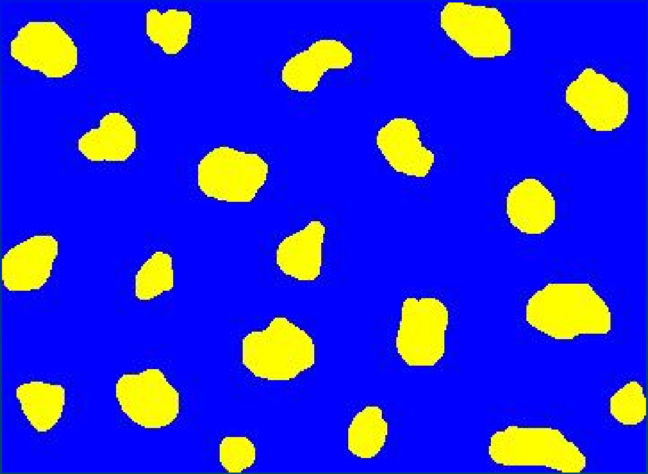

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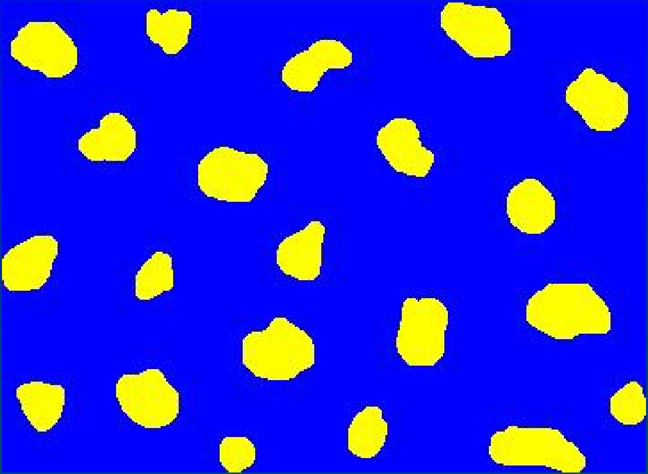

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Fluctuation boundaries

The most important condition expresses the fact that the ground state energy comes from those realizations of the potential that vanish on large sets:

(A4) Denote $E_0 := \inf \sigma(H_0) \leq \inf \sigma(H(\omega))$ and let

$$H_F := H_0 + \eta_{\max} \sum_{\alpha \in \mathcal{I}} U_\alpha ,$$

the subscript F standing for full coupling.
Assume that E_0 is a *fluctuation boundary* in the sense that

(i) $E_F := \inf \sigma(H_F) > E_0$, and
(ii) There is $m \in (0,2)$ and L^* such that for $m_d := 42 \cdot d$, all $L \geq L^*$ and $x \in \mathbb{Z}^d$

$$\mathbb{P}(\sigma(H^{\Lambda_L(x)}(\omega) \cap [E_0, E_0 + L^{-m}] \neq \emptyset) \leq L^{-m_d}.$$

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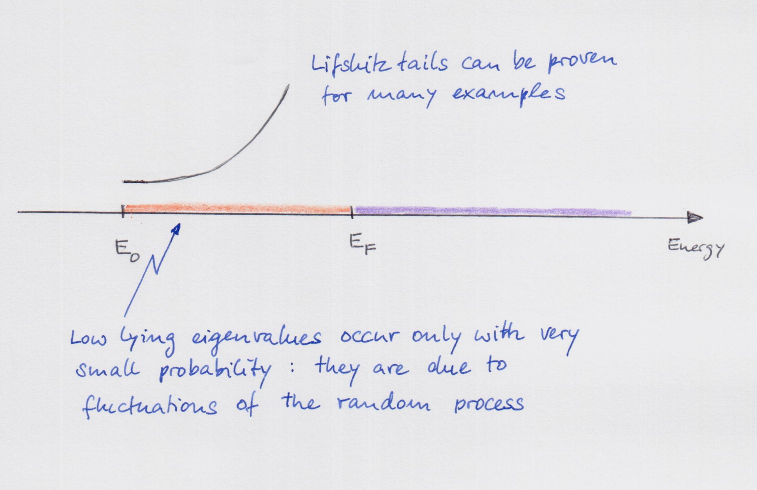

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The result

Our main result is

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Let $d \leq 3$ and assume (A1)-(A4). Then there exist $\delta > 0$, $0 < s < 1$, $\mu > 0$ and $C < \infty$ such that for $I := [E_0, E_0 + \delta]$, all open sets $G \subset \mathbb{R}^d$ and $x, y \in \mathbb{R}^d$,

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Exponential decay of fractional moments of the resolvent as described by (1) implies spectral and dynamical localization.

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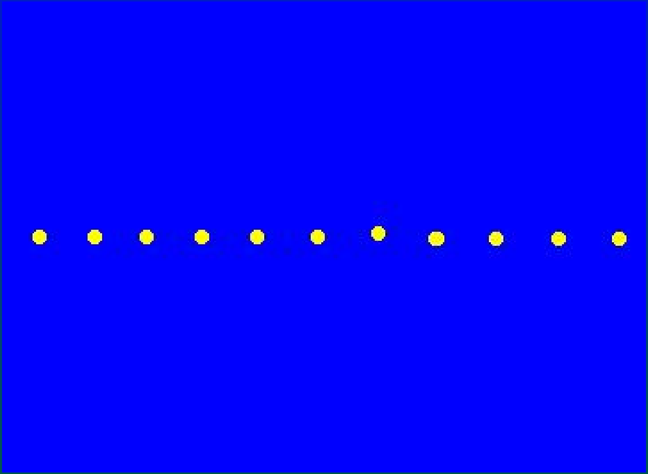
Surface potentials

The first application is concerned with surface potentials:

The necessary estimates at the fluctuation boundary was given by Kirsch and Warzel; we use an extension of their result:



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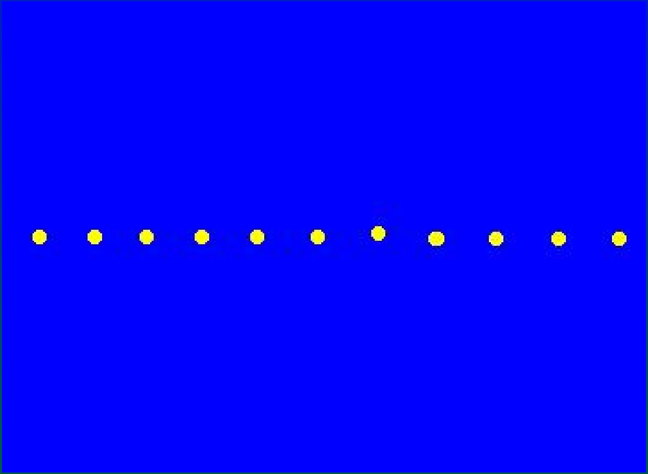

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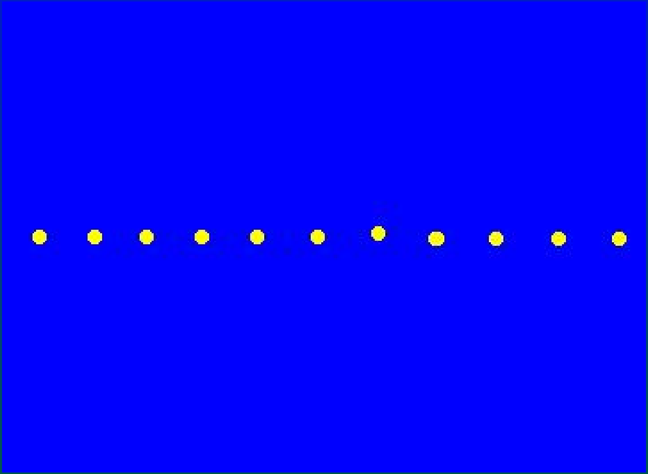

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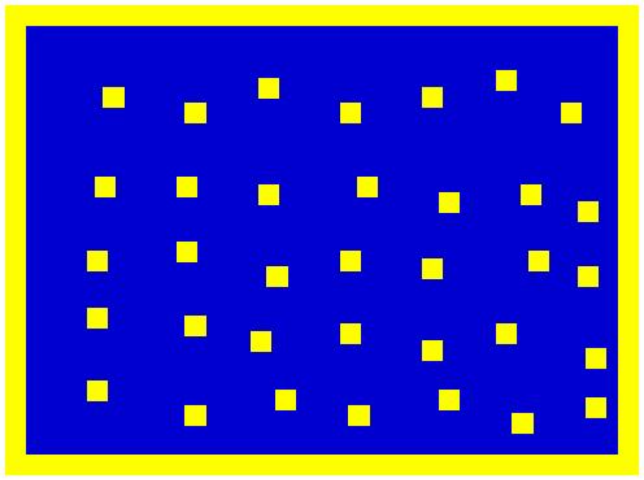

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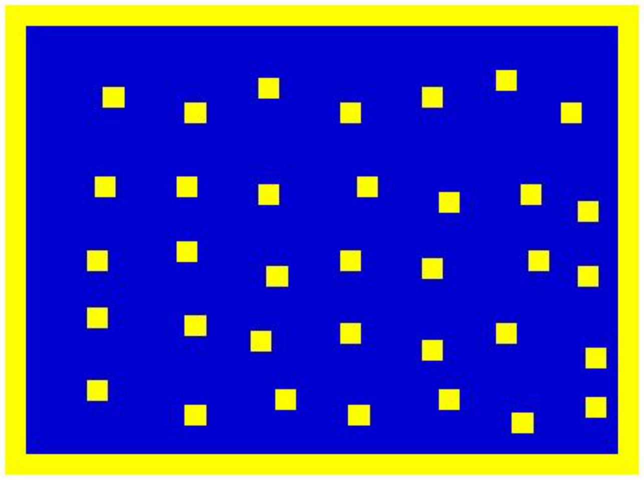

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