Random Operators: A Short (Hi)Story

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in honor of Robert Moody

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Outline

This is about random Schrödinger operators and their history.

- The Anderson model.
- The metal-insulator transition in spectral terms.
- The general formalism of random operators.
- Models with aperiodic order.

P.W. Anderson

In the celebrated article "Absence of Diffusion in Certain Random Lattices". *Phys. Rev.* **109** (5): 1492 –1505 from 1958, P.W. Anderson proposed a mathematical model to explain the phase transition from insulator to metal in disordered solids.

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from Anderson's Nobel lecture:

prising nonergodic behavior most clearly. I apologize for this brief excursion into mathematics, but please be assured that I include the least amount possible.

The first problem was to create a model which contained only essentials. This was simple enough: a linearized, random "tight-binding" model of non-interacting particles:

$$H = \sum_i E_i n_i + \sum_{ij} V_{ij} c_i^+ c_j \qquad (9)$$

in which the "hopping" integrals V_{ij} were taken to be nonrandom functions of r_{ij} (the sites i can sit on a lattice if we like) but E_i was chosen from a random probability distribution of width W (Fig. 14). The objects c_i could be harmonic

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The original Anderson model:

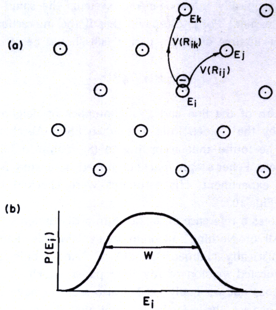


Fig. 14. Model for diffusion in a random lattice
a) sites and hopping integrals
b) probability distribution of E_i

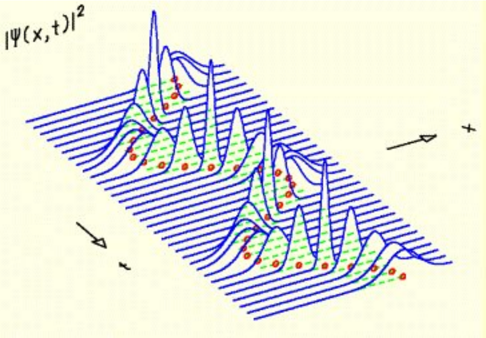
$$i\hbar \dot{c}_i^{+} = E_i c_i^{+} + \sum V_{ij} c_j^{+} \tag{10}$$

from Anderson's Nobel lecture:

But there is a very important fundamental truth about random systems we must always keep in mind : *no real atom is an average atom, nor is an experiment ever done on an ensemble of samples* . What we really need to know is the *probability distribution* of $\mathbf{Im}\sum$, *not* its average, because it's only each specific instance we are interested in. I would like to emphasize that this is the important, and deeply new, step taken here: the willingness to deal with *distributions*, not averages. Most of the recent progress in the fundamental physics of amorphous materials involves this same kind of step, which implies that a random system is to be treated not as just a dirty regular one, but in a fundamentally different way.

The Schrödinger equation.

$$\dot{\psi}(t) = -i(-\Delta + V)\psi(t), \psi(0) = \psi_0,$$



Quantum dynamics.

Easy: the solution of the Schrödinger equation is

$$\psi(t) = e^{-iHt}\psi_0.$$

But: what does it look like?

To this end one studies the spectral resolution of H.

- If ψ_0 is an eigenvector of H with eigenvalue $E_0 \Rightarrow$

  $$\psi(t) = e^{-iE_0 t}\psi_0.$$

  One speaks of a bound state.

- If the spectral measure $\rho^H_{\psi_0}$ of ψ_0 w.r.t H is continuous,

  $$\Rightarrow \lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T} \|\chi_B \psi(t)\|^2 dt = 0 \text{ for compact } B.$$

  One speaks of a scattering state.

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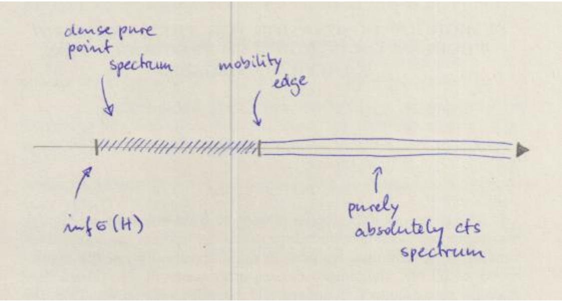
The metal-insulator transition in spectral terms.

- Atomic models are described by a negative potential V that decays rapidely enough near infinity:one gets bound states for negative energies and scattering states for positive energies.
- Solid states with perfect order are described by a periodic V $\Rightarrow H = -\Delta + V$ has only scattering states.
- P.W. Anderson proposed a new paradigm for disordered solids in dimension ≥ 3:

The metal-insulator transition in spectral terms.



Once translated into the language of spectral theory there is a transition from a

localized phase that exhibits pure point spectrum

(= only bound states = no transport)

to a

delocalized phase with absolutely continuous spectrum

(= scattering states = transport)

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The metal-insulator transition in spectral terms.

However, for genuine random models, there is no rigorous proof for the existence of a transition or even of the appearance of spectral components other than pure point, so far.
This is quite a strange situation: the unperturbed problem exhibits extended states and purely a.c. spectrum but for the perturbed one can prove the opposite spectral behavior only.

Anderson type models.

Here, disorder is modelled by random potentials:

$$H(\omega) = -\Delta + V_\omega$$

E.g., $\Omega = \mathbb{R}^{\mathbb{Z}^d}$, $\mathbb{P} = \mu^{\mathbb{Z}^d}$ a probability space describing independent, identically distributed coupling constants

$$V_\omega = \sum_{k \in \mathbb{Z}^d} \omega_k \cdot v(\cdot - k)$$

sometimes also called alloy type models. Localization has been proven for such models under additional technical hypotheses on μ and v.

History

Mathematical rigorous treatment needed

- new techniques
- led to new aspects of spectral theory ...
- ... with generic exotic spectral types

Localization: Goldsheidt, Molchanov, Pastur 1977 ...
Fröhlich, Spencer 1983,... with multiscale analysis
Books and surveys: in the list of references.

Covariant random operators

We start with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a measurable map

$$\Omega \ni \omega \mapsto H(\omega),$$

with values in the selfadjoint operators on a Hilbert space $\mathfrak{H}$. We call $H(\cdot)$ *covariant*, if there is an action of the group G (typically $\in \{\mathbb{Z}^d, \mathbb{R}^d\}$) on both Ω and $\mathfrak{H}$ s.t.

$$H(t \cdot \omega) = U_t^* H(\omega) U_t.$$

If the action of G on Ω is ergodic, then the spectra of $H(\omega)$ are ergodic, including the spectral types, i.e., there is a closed set Σ s.t.

$$\sigma(H(\omega)) = \Sigma \text{ for } \mathbb{P} - a.e.\omega.$$

Example: Anderson type

Here: $\Omega = \mathbb{R}^{\mathbb{Z}^d}$, $\mathbb{P} = \mu^{\mathbb{Z}^d}$

$$V_\omega = \sum_{k \in \mathbb{Z}^d} \omega_k \cdot v(\cdot - k),$$

$$H(\omega) = -\Delta + V_\omega$$

in $\mathfrak{H}$ (either $L^2(\mathbb{R}^d)$ or $\ell^2(\mathbb{Z}^d)$). Here G is $\mathbb{Z}^d$ and acts by translations.

Example: Poisson model

Here: Ω is the set of Radon measures on $\mathbb{R}^d$, $\mathbb{P}$ the measure of the Poisson point process,

$$V_\omega = v * \omega = \sum_{k \in \mathbb{N}} v(\cdot - X_k(\omega)),$$

$$H(\omega) = -\Delta + V_\omega$$

in $\mathfrak{H} = L^2(\mathbb{R}^d)$. Here G is $\mathbb{R}^d$ and it acts by translations.

Periodic and almost periodic operators

... can also be viewed in this framework. In the periodic case, $\Omega = \mathbb{T}^d$ endowed with the normalised Lebesgue measure. For the almost periodic case, we mention the most important example, the **almost Mathieu operator**:

$$H_{\alpha,\lambda,\omega} = \Delta + 2\lambda \cos(2\pi\alpha n + \omega) \text{ on } \ell^2(\mathbb{Z}).$$

An enormous amount of work has been devoted to various aspects of the spectral analysis of this model for different parameter regions. Suffice it to say that all spectral types occur and that the spectrum is a Cantor set for $\lambda \neq 0$ and irrational α (10 Martini problem).

Operator algebras

The almost Mathieu operator is strongly related to the irrational rotation algebra $A_\alpha = C(\mathbb{T}) \ltimes_\sigma \mathbb{Z}$.
More generally, random operators can be investigated in terms of noncommutative geometry, e.g., **gap labelling**. See work of Bellissard and coworkers.
The important **integrated density of states** of random Schrödinger operators can then be viewed as the trace (in certain von Neumann algebras) of the spectral projection.

Subshifts

Here, Ω is a compact, translationinvariant subset of $\mathcal{A}^{\mathbb{Z}}$ and $\mathbb{Z}$ acts via translations. In the topological category, one is typically dealing with **minimal dynamical systems**. Quite often one also assumes that an ergodic probability measure is given on Ω. An important example is the Fibonacci sequence (resp. its hull). The general expectation is that these operators exhibit purely singular continuous spectrum, the spectrum as a set being a Cantor set of measure zero. See the local experts :-)

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Delonistic potentials

... are those of the type

$$V_\omega(\cdot) = \sum_{t\in\omega} v(\cdot - t)$$

where ω is a Delone set and v a single site potential. In joint work with D. Lenz we show that singular continuous spectrum is generic in a suitable space of Delone sets. In joint work with S. Klassert and D. Lenz we show purely singular continuous spectrum for certain aperiodic Delone dynamical systems.

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