Analysis II, 15. Übung

SS 2014 http://www.tu-chemnitz.de/~peju/lehre/ana.html

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1. Man bestimme mit Hilfe (elementarer) Zurückführung auf Grundintegrale

   (a) $\displaystyle\int \left(1-\frac{1}{x^2}\right)\sqrt{x\sqrt{x}}\,dx$, (b) $\displaystyle\int \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}}\,dx$, (c) $\displaystyle\int \sqrt{1-\sin 2x}\,dx$,

   (d) $\displaystyle\int \frac{e^{3x}+1}{e^x+1}\,dx$, (e) $\displaystyle\int \tan^2 x\,dx$, (f) **(HA)** $\displaystyle\int (1-x)(1-2x)(1-3x)\,dx$,

   (g) **(HA)** $\displaystyle\int \frac{x+1}{\sqrt{x}}\,dx$, (h) **(HA)** $\displaystyle\int \frac{\sqrt{x}-2\sqrt[3]{x^2}+1}{\sqrt[4]{x}}\,dx$, (i) **(HA)** $\displaystyle\int \frac{(1-x)^3}{x\sqrt[3]{x}}\,dx$.

2. Man bestimme mit Hilfe geeigneter Substitutionen

   (a) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$, (b) $\displaystyle\int \frac{1}{x^2}\,\sin\frac{1}{x}\,dx$, (c) $\displaystyle\int \frac{dx}{\sqrt{x(1+x)}}$ (d) $\displaystyle\int \frac{dx}{(x\ln x)\,\ln(\ln x)}$,

   (e) $\displaystyle\int \tan x\,dx$, (f) $\displaystyle\int \sin^5 x\cos x\,dx$, (g) $\displaystyle\int \frac{dx}{\sin x}$ (h) **(HA)** $\displaystyle\int \frac{\arctan x}{1+x^2}\,dx$,

   (i) **(HA)** $\displaystyle\int \frac{x\,dx}{3-2x^2}$, (j) **(HA)** $\displaystyle\int \frac{e^x}{2+e^x}\,dx$, (k) **(HA)** $\displaystyle\int x\,e^{-x^2}\,dx$,

   (l) **(HA)** $\displaystyle\int \frac{dx}{\sinh x}$.

3. Man bestimme mittels partieller Integration

   (a) $\displaystyle\int \left(\frac{\ln x}{x}\right)^2 dx$, (b) $\displaystyle\int \sqrt{x}\ln^2 x\,dx$, (c) $\displaystyle\int \arctan\sqrt{x}\,dx$,

   (d) $\displaystyle\int \sin x\,\ln(\tan x)\,dx$, (e) $\displaystyle\int x\ln\frac{1+x}{1-x}\,dx$, (f) $\displaystyle\int \sin^2 x\,dx$,

   (g) **(HA)** $\displaystyle\int e^{\alpha x}\sin\beta x\,dx$ $(\alpha,\beta\in\mathbb{R},\ \beta\neq 0)$, (h) **(HA)** $\displaystyle\int x^2 e^{-2x}\,dx$,

   (i) **(HA)** $\displaystyle\int x^2\sin 2x\,dx$, (j) **(HA)** $\displaystyle\int \arctan x\,dx$, **(Z)** $\displaystyle\int x^n e^x\,dx$.

4. Man berechne mittels Partialbruchzerlegung

   (a) $\displaystyle\int \frac{2x+3}{(x-2)(x+5)}\,dx$, (b) $\displaystyle\int \frac{x^4dx}{x^4+5x^2+4}$, (c) $\displaystyle\int \frac{dx}{(x+1)(x+2)^2(x+3)^2}$,

   (d) $\displaystyle\int \left(\frac{x}{x^2+3x+2}\right)^2 dx$, (e) $\displaystyle\int \frac{dx}{x^5+x^4-2x^3-2x^2+x+1}$, (f) $\displaystyle\int \frac{dx}{x^4+1}$,

   (g) **(HA)** $\displaystyle\int \frac{x\,dx}{(x+1)(x+2)(x+3)}$, (h) **(HA)** $\displaystyle\int \left(\frac{x}{x^2-3x+2}\right)^2 dx$,

   (i) **(HA)** $\displaystyle\int \frac{dx}{x^3+1}$, (j) **(HA)** $\displaystyle\int \frac{x^2\,dx}{(x^2+2x+2)^2}$.

5. Man berechne

   (a) $\displaystyle\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx$, (b) $\displaystyle\int \frac{dx}{1+2\sin^2 x}$, (c) $\displaystyle\int \frac{dx}{(x^2-1)(x^2+1)}$,

   (d) $\displaystyle\int \frac{1+x}{1-x}\,dx$, (e) $\displaystyle\int \frac{2x+5}{x^2+4x+5}\,dx$, (f) $\displaystyle\int \frac{x^2}{(x-1)^{100}}\,dx$, (l) **(HA)** $\displaystyle\int \frac{dx}{2\sin 2x}$,

   (g) $\displaystyle\int \frac{dx}{(x^2-4x+4)(x^2-4x+5)}$, (h) $\displaystyle\int \sqrt{\frac{a+x}{a-x}}dx$, (i) **(HA)** $\displaystyle\int (\arcsin x)^2\,dx$,

   (j) **(HA)** $\displaystyle\int x\,(\arctan x)^2\,dx$, (k) **(HA)** $\displaystyle\int \frac{x\,dx}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}$.

6. Entwickeln Sie eine Rekursionsformel zur Berechnung von

   (a) $I_n(x) = \int \frac{dx}{\cos^n x}$, (b) **(HA)** $S_n(x) = \int \sin^n x dx$.

(Z1) Berechnen Sie

   (a) $\int \frac{\cos x}{\sqrt{\cos 2x}}\, dx$, (b) $\int \frac{x \arctan x}{\sqrt{1-x^2}}\, dx$, (c) $\int (\tan x)\, \tan(x+a)\, dx$,

   (d) $\int x f''(x)\, dx$, (e) $\int \frac{(9\sin^2 x - 3\sin^3 x)\cos x - 5\sin 2x + 10\cos x}{\sin^4 x - 2\sin^3 x + 2\sin x - 1}\, dx$,

   (f) $\int \frac{\cos^2 x}{1+\sin^2 x}\, dx$, (g) $\int \frac{dx}{1+\sin x + \cos x}$.

7. Berechnen Sie die Fläche, die von den Graphen der Funktionen $f(x) = 2 - x^2$ und $g(x) = x^2 - 2x + \frac{3}{2}$ eingeschlossen wird.

8. Berechnen Sie

   (a) $\int_{-1}^{1} x|x|\, dx$, (b) $\int_{2}^{3,5} \frac{dx}{\sqrt{5+4x-x^2}}$, (c) $\int_0^1 \frac{x\, dx}{a+bx}$, (d) $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx$.

9. Zeigen Sie, dass für eine stetige Funktion $f : [0,1] \longrightarrow \mathbb{R}$ gilt

   (a) $\int_0^{\frac{\pi}{2}} f(\sin x)\, dx = \int_0^{\frac{\pi}{2}} f(\cos x)\, dx$, (b) $\int_0^{\pi} x f(\sin x)\, dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x)\, dx$.

10. Beweisen Sie folgende Abschätzungen:

    (a) $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \frac{\pi}{6}$ (b) $\frac{1}{2} < \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x}{x}\, dx < \frac{\sqrt{2}}{2}$ (b') $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x}{x}\, dx < \ln 2$

    (Z) $0 < \int_{100\pi}^{200\pi} \frac{\cos x}{x}\, dx < \frac{1}{100\pi}$

11. Bestimmen Sie

    (a) $\lim_{x\to\infty} \frac{1}{x} \int_a^x \arctan y\, dy$, (b) $\lim_{x\to\infty} \int_0^x \frac{\sqrt{1+t^4}\, dt}{x^3}$, (c) $\lim_{x\to 0} x \int_x^1 \frac{\cos t}{t^2}\, dt$.

12. Suppose that $f : [0,\infty) \longrightarrow (0,\infty)$ is continuous. Show that the function

$$\varphi : [0,\infty) \longrightarrow \mathbb{R}, \quad x \mapsto \frac{\int_0^x t\, f(t)\, dt}{\int_0^x f(t) dt}$$

    is monotonically increasing.

(Z2) Es sei $f : [a,b] \longrightarrow \mathbb{R}$ beschränkt und konkav. Man zeige, dass dann

$$(b-a)\, \frac{f(a)+f(b)}{2} \le \int_a^b f(x)\, dx \le (b-a)\, f\left(\frac{a+b}{2}\right)$$

gilt.