Aufgabe 22.23

Lösen Sie die Anfangswertaufgabe

$$\begin{aligned} \dot{x} &= 2x-2y-\ \ z \\ \dot{y} &= 3x-5y-3z \\ \dot{z} &= 2x-4y-\ \ z \\ x(0) &= 2,\ y(0)=-2,\ z(0)=-1 \quad ! \end{aligned}$$

Lösung:

$$\begin{vmatrix} 2-\lambda & -2 & -1 \\ 3 & -5-\lambda & -3 \\ 2 & -4 & -1-\lambda \end{vmatrix} \begin{aligned} &= (2-\lambda)(5+\lambda)(1+\lambda)+12+12-2(5+\lambda)-12(2-\lambda)-6(1+\lambda) \\ &= (2-\lambda)(5+6\lambda+\lambda^2)+24-10-2\lambda-24+12\lambda-6-6\lambda \\ &= 10+12\lambda+2\lambda^2-5\lambda-6\lambda^2-\lambda^3-16+4\lambda = -\lambda^3-4\lambda^2+11\lambda-6 \\ &= 0 \end{aligned}$$

$\lambda^3+4\lambda^2-11\lambda+6=0$, $\lambda_1=1$ offensichtlich, $(\lambda^3+4\lambda^2-11\lambda+6):(\lambda-1)=\lambda^2+5\lambda-6,$

$$\begin{array}{l} \lambda^3\ -\lambda^2 \\ \hline \quad 5\lambda^2-11\lambda+6 \\ \quad 5\lambda^2\ -5\lambda \\ \hline \qquad\quad -6\lambda+6 \\ \qquad\quad -6\lambda+6 \\ \hline \qquad\qquad\quad 0 \end{array}$$

$$\lambda_{2/3} = -\frac{5}{2} \pm \sqrt{\frac{25}{4}+\frac{24}{4}} = \frac{5}{2} \pm \frac{7}{2} = 1,-6$$

zu $\lambda_{1/2}=1$:

$$\begin{array}{rrr} 1 & -2 & -1 \\ 3 & -6 & -3 \\ 2 & -4 & -2 \\ \hline 1 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$$

Eigenvektoren $A\begin{pmatrix}2\\1\\0\end{pmatrix}$, $B\begin{pmatrix}1\\0\\1\end{pmatrix}$

zu $\lambda_3=-6$:

$$\begin{array}{rrr} 8 & -2 & -1 \\ 3 & 1 & -3 \\ 2 & -4 & 5 \\ \hline 1 & -2 & \frac{5}{2} \\ 3 & 1 & -3 \\ 8 & -2 & -1 \\ \hline 1 & -2 & \frac{5}{2} \\ 0 & 7 & -\frac{21}{2} \\ 0 & 14 & -21 \\ \hline 1 & -2 & \frac{5}{2} \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 \\ \hline 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 \end{array}$$

Eigenvektor $\widetilde{C}\begin{pmatrix}\frac{1}{2}\\\frac{3}{2}\\1\end{pmatrix} = C\begin{pmatrix}1\\3\\2\end{pmatrix}$

$$\vec{x}(t) = A\begin{pmatrix}2\\1\\0\end{pmatrix}\mathrm{e}^t + B\begin{pmatrix}1\\0\\1\end{pmatrix}\mathrm{e}^t + C\begin{pmatrix}1\\3\\2\end{pmatrix}\mathrm{e}^{-6t},\quad \vec{x}(0) = A\begin{pmatrix}2\\1\\0\end{pmatrix} + B\begin{pmatrix}1\\0\\1\end{pmatrix} + C\begin{pmatrix}1\\3\\2\end{pmatrix} = \begin{pmatrix}2\\-2\\-1\end{pmatrix}$$

$$\begin{array}{rrr|r} 2 & 1 & 1 & 2 \\ 1 & 0 & 3 & -2 \\ 0 & 1 & 2 & -1 \\ \hline 1 & 0 & 3 & -2 \\ 0 & 1 & 2 & -1 \\ 2 & 1 & 1 & 2 \end{array} \qquad \begin{array}{rrr|r} 1 & 0 & 3 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & -5 & 6 \\ \hline 1 & 0 & 3 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -7 & 7 \end{array} \qquad \begin{array}{rrr|r} 1 & 0 & 3 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}$$

$$\underline{\underline{\vec{x}(t)}} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \mathrm{e}^{t} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \mathrm{e}^{t} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \mathrm{e}^{-6t} = \underline{\underline{\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \mathrm{e}^{t} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \mathrm{e}^{-6t}}}$$