Aufgabe 19.27

Sei U ein Skalar- und $\vec{v}$ ein Vektorfeld. Beweisen Sie die „Produktregel“
$\operatorname{div}(U\vec{v}) = \operatorname{grad} U \cdot \vec{v} + U \operatorname{div} \vec{v}$!

Lösung:

$$\operatorname{div}(U\vec{v}) = \operatorname{div}\left(U\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) = \operatorname{div}\begin{pmatrix} Uv_1 \\ Uv_2 \\ Uv_3 \end{pmatrix} = \frac{\partial\,(Uv_1)}{\partial x} + \frac{\partial\,(Uv_2)}{\partial y} + \frac{\partial\,(Uv_3)}{\partial z}$$

$$= \frac{\partial U}{\partial x}v_1 + U\frac{\partial v_1}{\partial x} + \frac{\partial U}{\partial y}v_2 + U\frac{\partial v_2}{\partial y} + \frac{\partial U}{\partial z}v_3 + U\frac{\partial v_3}{\partial z}$$

$$= \frac{\partial U}{\partial x}v_1 + \frac{\partial U}{\partial y}v_2 + \frac{\partial U}{\partial z}v_3 + U\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) = \operatorname{grad} U \cdot \vec{v} + U \operatorname{div} \vec{v} \quad \text{qed.}$$