Aufgabe 19.17

Sei $\vec{x} \in \mathbb{R}^3$. Berechnen Sie $\operatorname{div}\operatorname{grad}\dfrac{1}{\|\vec{x}\|}$!

Lösung:

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \|\vec{x}\| = \sqrt{x^2+y^2+z^2}, \quad \frac{1}{\|\vec{x}\|} = \left(x^2+y^2+z^2\right)^{-1/2},$$

$$\operatorname{grad}\frac{1}{\|\vec{x}\|} = -\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}\begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = \begin{pmatrix} -\dfrac{x}{\left(x^2+y^2+z^2\right)^{3/2}} \\ -\dfrac{y}{\left(x^2+y^2+z^2\right)^{3/2}} \\ -\dfrac{z}{\left(x^2+y^2+z^2\right)^{3/2}} \end{pmatrix}$$

$$\begin{aligned}
\operatorname{div}\operatorname{grad}\frac{1}{\|\vec{x}\|} &= -\frac{\partial}{\partial x}\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}} - \frac{\partial}{\partial y}\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}} - \frac{\partial}{\partial z}\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}} \\
&= -\frac{\left(x^2+y^2+z^2\right)^{3/2} - x\,\frac{3}{2}\left(x^2+y^2+z^2\right)^{1/2}2x}{\left(x^2+y^2+z^2\right)^3} \\
&\quad -\frac{\left(x^2+y^2+z^2\right)^{3/2} - y\,\frac{3}{2}\left(x^2+y^2+z^2\right)^{1/2}2y}{\left(x^2+y^2+z^2\right)^3} \\
&\quad -\frac{\left(x^2+y^2+z^2\right)^{3/2} - z\,\frac{3}{2}\left(x^2+y^2+z^2\right)^{1/2}2z}{\left(x^2+y^2+z^2\right)^3} \\
&= -\frac{\left(x^2+y^2+z^2\right) - 3x^2}{\left(x^2+y^2+z^2\right)^{5/2}} - \frac{\left(x^2+y^2+z^2\right) - 3y^2}{\left(x^2+y^2+z^2\right)^{5/2}} - \frac{\left(x^2+y^2+z^2\right) - 3z^2}{\left(x^2+y^2+z^2\right)^{5/2}} \\
&= -\frac{3\left(x^2+y^2+z^2\right) - 3x^2 - 3y^2 - 3z^2}{\left(x^2+y^2+z^2\right)^{5/2}} = \underline{\underline{0}}
\end{aligned}$$