Aufgabe 16.14

Berechnen Sie die Eigenwerte und Eigenvektoren folgender Matrizen:

a) $\begin{pmatrix} 4 & -2 \\ 3 & -3 \end{pmatrix}$, b) $\begin{pmatrix} 4 & 37 \\ -\frac{1}{2} & -3 \end{pmatrix}$, c) $\begin{pmatrix} 8 & 9 & 2 \\ 1 & 0 & -2 \\ -8 & -2 & 7 \end{pmatrix}$!

Lösung:

a) Eigenwerte: $\det(\boldsymbol{A}-\lambda\boldsymbol{E}) = \begin{vmatrix} 4-\lambda & -2 \\ 3 & -3-\lambda \end{vmatrix} = (4-\lambda)(-3-\lambda)+6 = \lambda^2-\lambda-6 = 0$

$$\Longrightarrow \quad \lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4}+\frac{24}{4}} = \begin{cases} 3 \\ -2 \end{cases}$$

Eigenvektor zu $\lambda_1 = 3$:

$$\begin{array}{rr} 1 & -2 \\ 3 & -6 \\ \hline 1 & -2 \end{array} \qquad x_1-2x_2=0,\ x_1=2x_2,\quad x_2=C,\ x_1=2C \qquad \text{EV } C\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Eigenvektor zu $\lambda_2 = -2$:

$$\begin{array}{rr} 6 & -2 \\ 3 & -1 \\ \hline -3 & 1 \end{array} \qquad -3x_1+x_2=0,\ x_2=3x_2,\quad x_1=C,\ x_2=3C \qquad \text{EV } D\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

b) Eigenwerte:

$$\det(\boldsymbol{A}-\lambda\boldsymbol{E}) = \begin{vmatrix} 4-\lambda & 37 \\ -\frac{1}{2} & -3-\lambda \end{vmatrix} = (4-\lambda)(-3-\lambda)+\frac{37}{2} = \lambda^2-\lambda-12+\frac{37}{2} = \lambda^2-\lambda+\frac{13}{2} = 0$$

$$\Longrightarrow \quad \lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4}-\frac{26}{4}} = \frac{1}{2} \pm \frac{5}{2}\mathrm{i}$$

Eigenvektor zu $\lambda_1 = \frac{1}{2}+\frac{5}{2}\mathrm{i}$:

$$\begin{array}{cc} \frac{7}{2}-\frac{5}{2}\mathrm{i} & 37 \\ -\frac{1}{2} & -\frac{7}{2}-\frac{5}{2}\mathrm{i} \\ \hline 7-5\mathrm{i} & 74 \\ -1 & -7-5\mathrm{i} \\ \hline 1 & 7+5\mathrm{i} \\ 7-5\mathrm{i} & 74 \\ \hline 1 & 7+5\mathrm{i} \\ 0 & 74-(7-5\mathrm{i})(7+5\mathrm{i})=0 \\ \hline 1 & 7+5\mathrm{i} \end{array} \qquad x_1+(7+5\mathrm{i})x_2=0,\quad x_1=(-7-5\mathrm{i})x_2 \qquad \text{EV } C\begin{pmatrix} -7-5\mathrm{i} \\ 1 \end{pmatrix}$$

Eigenvektor zu $\lambda_2 = \frac{1}{2}-\frac{5}{2}\mathrm{i}$:

$$\begin{array}{cc} \frac{7}{2}+\frac{5}{2}\mathrm{i} & 37 \\ -\frac{1}{2} & -\frac{7}{2}+\frac{5}{2}\mathrm{i} \\ \hline 7+5\mathrm{i} & 74 \\ -1 & -7+5\mathrm{i} \\ \hline 1 & 7-5\mathrm{i} \\ 7+5\mathrm{i} & 74 \\ \hline 1 & 7-5\mathrm{i} \\ 0 & 74-(7+5\mathrm{i})(7-5\mathrm{i})=0 \\ \hline 1 & 7-5\mathrm{i} \end{array} \qquad x_1+(7-5\mathrm{i})x_2=0,\quad x_1=(-7+5\mathrm{i})x_2 \qquad \text{EV } D\begin{pmatrix} -7+5\mathrm{i} \\ 1 \end{pmatrix}$$

c)
$$\begin{vmatrix} 8-\lambda & 9 & 2 \\ 1 & -\lambda & -2 \\ -8 & -2 & 7-\lambda \end{vmatrix} \begin{aligned} &= -\lambda(8-\lambda)(7-\lambda)+144-4-16\lambda-9(7-\lambda)-4(8-\lambda) \\ &= -\lambda(\lambda^2-15\lambda+56)+140-16\lambda-63+9\lambda-32+4\lambda \\ &= -\lambda^3+15\lambda^2-56\lambda+45-\lambda = -\lambda^3+15\lambda^2-59\lambda+45 = 0 \end{aligned}$$

Offensichtlich ist $\lambda_1=1$ eine Nullstelle des charakteristischen Polynoms.

$(\lambda^3-15\lambda^2+59\lambda-45):(\lambda-1) = \lambda^2-14\lambda+45, \quad \lambda_{2/3}=7\pm\sqrt{49-45}=\begin{cases} 5 \\ 9 \end{cases}$

$$\begin{array}{l} \lambda^3-\ \ \lambda^2 \\ \hline \quad -14\lambda^2+59\lambda-45 \\ \quad -14\lambda^2+14\lambda \\ \hline \qquad\qquad 45\lambda-45 \\ \qquad\qquad 45\lambda-45 \\ \hline \qquad\qquad\qquad 0 \end{array}$$

EV zu EW $\lambda_1 = 1$:

$$\begin{array}{rrr}
7 & 9 & 2 \\
1 & -1 & -2 \\
-8 & -2 & 6 \\ \hline
1 & -1 & -2 \\
7 & 9 & 2 \\
-8 & -2 & 6 \\ \hline
1 & -1 & -1 \\
0 & 16 & 16 \\
0 & -10 & -10 \\ \hline
1 & -1 & -2 \\
0 & 1 & 1 \\ \hline
1 & 0 & -1 \\
0 & 1 & 1
\end{array}$$

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

EV $C \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

EV zu EW $\lambda_2 = 5$:

$$\begin{array}{rrr}
3 & 9 & 2 \\
1 & -5 & -2 \\
-8 & -2 & 2 \\ \hline
1 & -5 & -2 \\
3 & 9 & 2 \\
-8 & -2 & 2 \\ \hline
1 & -5 & -2 \\
0 & 24 & 8 \\
0 & -42 & -14 \\ \hline
1 & -5 & -2 \\
0 & 3 & 1 \\ \hline
1 & 1 & 0 \\
0 & 3 & 1
\end{array}$$

$$\begin{aligned} x_1 + x_2 &= 0 \\ 3x_2 + x_3 &= 0 \end{aligned}$$

EV $D \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

EV zu EW $\lambda_3 = 9$:

$$\begin{array}{rrr}
-1 & 9 & 2 \\
1 & -9 & 2 \\
-8 & -2 & -2 \\ \hline
1 & -9 & -2 \\
-1 & 9 & 2 \\
-8 & -2 & -2 \\ \hline
1 & -9 & -2 \\
0 & 0 & 0 \\
0 & -74 & -18 \\ \hline
1 & -9 & -2 \\
0 & \frac{37}{9} & 1 \\ \hline
1 & -\frac{7}{9} & 0 \\
0 & \frac{37}{9} & 1
\end{array}$$

$$\begin{aligned} x_1 - \tfrac{7}{9}x_2 &= 0 \\ \tfrac{37}{9}x_2 + \quad x_3 &= 0 \end{aligned}$$

EV $\widetilde{E} \begin{pmatrix} \frac{7}{9} \\ 1 \\ -\frac{37}{9} \end{pmatrix} = E \begin{pmatrix} 7 \\ 9 \\ -37 \end{pmatrix}$