Aufgabe 14.37

Ermitteln Sie die Fourierreihe von $f(x)=\begin{cases} 0, & -\pi\le x\le 0 \\ \sin x, & 0\le x\le\pi \end{cases}$ in reeller und komplexer Form!

Lösung:
Periodenlänge 2π, $\quad f(x)=\dfrac{a_0}{2}+\sum\limits_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)$ (da $f(x)$ überall stetig),

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\cos kx\,\mathrm{d}x,\ \ b_k=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin kx\,\mathrm{d}x$$

Formeln: $\sin\alpha\sin\beta=\frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta))$, $\quad\sin\alpha\cos\beta=\frac{1}{2}(\sin(\alpha-\beta)+\sin(\alpha+\beta))$

$$a_k=\frac{1}{\pi}\int_0^{\pi}\sin x\,\cos kx\,\mathrm{d}x=\frac{1}{2\pi}\int_0^{\pi}\big(\sin(1-k)x+\sin(1+k)x\big)\mathrm{d}x=\frac{1}{2\pi}\left[\frac{\cos(1-k)x}{k-1}-\frac{\cos(1+k)x}{k+1}\right]_0^{\pi}$$

$$=\frac{1}{2\pi}\left(\frac{(-1)^{1-k}-1}{k-1}-\frac{(-1)^{k+1}-1}{k+1}\right)=\begin{cases}\dfrac{1}{2\pi}\left(\dfrac{-2}{k-1}-\dfrac{-2}{k+1}\right), & k \text{ gerade}\\ 0, & k \text{ ungerade}, \neq 1\end{cases}$$

$$=\begin{cases}-\dfrac{1}{\pi}\,\dfrac{k+1-k+1}{k^2-1}=-\dfrac{2}{\pi(k^2-1)}, & k \text{ gerade}\\ 0, & k \text{ ungerade}, \neq 1\end{cases}$$

$$a_1=\frac{1}{\pi}\int_0^{\pi}\sin x\,\cos x\,\mathrm{d}x=\frac{1}{2\pi}\int_0^{\pi}\sin 2x\,\mathrm{d}x=-\frac{\cos 2x}{4\pi}\Big|_0^{\pi}=0$$

$$b_k=\frac{1}{\pi}\int_0^{\pi}\sin x\,\sin kx\,\mathrm{d}x=\frac{1}{2\pi}\int_0^{\pi}\big(\cos(1-k)x-\cos(1+k)x\big)\mathrm{d}x=\frac{1}{2\pi}\left[\frac{\sin(1-k)x}{1-k}-\frac{\sin(1+k)x}{1+k}\right]_0^{\pi}$$
$$=0,\quad k\neq 1$$

$$b_1=\frac{1}{\pi}\int_0^{\pi}\sin x\,\sin x\,\mathrm{d}x=\frac{1}{2\pi}\int_0^{\pi}(\cos 0x-\cos 2x)\,\mathrm{d}x=\frac{1}{2\pi}\int_0^{\pi}(1-\cos 2x)\,\mathrm{d}x=\frac{1}{2\pi}\left[x-\frac{\sin 2x}{2}\right]_0^{\pi}=\frac{1}{2}$$

Reelle Form der Fourierreihe:

$$f(x)=\frac{1}{\pi}+\frac{1}{2}\sin x-\frac{2}{\pi}\sum_{l=1}^{\infty}\frac{\cos 2lx}{(2l-1)(2l+1)}=\frac{1}{\pi}+\frac{1}{2}\sin x-\frac{2}{\pi}\left(\frac{\cos 2x}{1\cdot 3}+\frac{\cos 4x}{3\cdot 5}+\frac{\cos 6x}{5\cdot 7}+\ldots\right)$$

Komplexe Form der Fourierreihe: Mit $\cos x=\dfrac{\mathrm{e}^{\mathrm{i}x}+\mathrm{e}^{-\mathrm{i}x}}{2}$, $\sin x=\dfrac{\mathrm{e}^{\mathrm{i}x}-\mathrm{e}^{-\mathrm{i}x}}{2\mathrm{i}}$ erhält man

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)=\sum_{k=-\infty}^{\infty}c_k\mathrm{e}^{\mathrm{i}kx}\ \text{ mit } c_0=\frac{a_0}{2},\ c_k=\frac{a_k-\mathrm{i}b_k}{2},\ c_{-k}=\frac{a_k+\mathrm{i}b_k}{2}\ (k=1,2,\ldots).$$

$$a_0=\frac{2}{\pi},\ c_0=\frac{1}{\pi},\ c_1=-\frac{\mathrm{i}}{4},\ c_{-1}=\frac{\mathrm{i}}{4},\quad c_{2l}=c_{-2l}=-\frac{1}{\pi\big((2l)^2-1\big)},\ c_{2l+1}=c_{-2l-1}=0\ \ (l=1,2,\ldots)$$

Also gilt $f(x)=-\dfrac{\mathrm{i}}{4}\mathrm{e}^{\mathrm{i}x}+\dfrac{\mathrm{i}}{4}\mathrm{e}^{-\mathrm{i}x}-\dfrac{1}{\pi}\sum\limits_{l=-\infty}^{\infty}\dfrac{1}{\big((2l)^2-1\big)}\mathrm{e}^{\mathrm{i}2lx}$ $\quad\big(c_0=\dfrac{1}{\pi}$ ergibt sich für $l=0.\big)$.