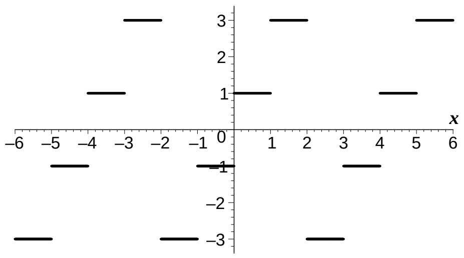
Aufgabe 14.30

Die Funktion $f(x)=\begin{cases}-3, & -2<x\leq -1\\ -1, & -1<x\leq 0\\ 1, & 0<x\leq 1\\ 3, & 1<x\leq 2\end{cases}$ werde 4-periodisch fortgesetzt.

a) Skizzieren Sie die Funktion!
b) Entwickeln Sie die Funktion in eine Fourierreihe!
c) Gegen welche Funktion konvergiert die Fourierreihe?

Lösung:

a)



b) Periodenlänge $T=4,\quad f(x)=\sim\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos\frac{k\pi}{2}x+b_k\sin\frac{k\pi}{2}x$

$a_k=0,\ k=0,1,2,\ldots,$ da $f(x)$ ungerade

$$b_k=\frac{1}{2}\int_{-2}^{2}f(x)\sin\frac{k\pi}{2}x\,\mathrm{d}x=\int_{0}^{2}f(x)\sin\frac{k\pi}{2}x\,\mathrm{d}x=\int_{0}^{1}\sin\frac{k\pi}{2}x\,\mathrm{d}x+3\int_{1}^{2}\sin\frac{k\pi}{2}x\,\mathrm{d}x$$

$$=\left[\frac{-\cos\frac{k\pi}{2}x}{\frac{k\pi}{2}}\right]_0^1-\left[\frac{3\cos\frac{k\pi}{2}x}{\frac{k\pi}{2}}\right]_1^2=\frac{-\cos\frac{k\pi}{2}+1}{\frac{k\pi}{2}}-3\frac{\cos k\pi-\cos\frac{k\pi}{2}}{\frac{k\pi}{2}}$$

k ungerade: $b_k=\frac{1}{\frac{k\pi}{2}}-3\frac{-1}{\frac{k\pi}{2}}=\frac{4}{\frac{k\pi}{2}}=\frac{8}{k\pi}$

$k=4n:\qquad b_k=\frac{-1+1}{\frac{k\pi}{2}}-3\frac{1-1}{\frac{k\pi}{2}}=0$

$k=4n+2:\ b_k=\frac{-(-1)+1}{\frac{k\pi}{2}}-3\frac{1-(-1)}{\frac{k\pi}{2}}=\frac{2-6}{\frac{k\pi}{2}}=-\frac{8}{k\pi}$

$$f(x)\sim\frac{8}{\pi}\left(\sin\frac{\pi}{2}x-\frac{\sin 2\frac{\pi}{2}x}{2}+\frac{\sin 3\frac{\pi}{2}x}{3}+\frac{\sin 5\frac{\pi}{2}x}{5}-\frac{\sin 6\frac{\pi}{2}x}{6}+\frac{\sin 7\frac{\pi}{2}x}{7}\pm\cdots\right)$$

$$=\frac{8}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\frac{\pi}{2}x}{2n+1}-\frac{8}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi x}{4n+2}$$

c) Konvergenz gegen $\begin{cases}f(x), & x\notin\mathbb{Z}\\ 0, & x=2l\\ 2, & x=4l+1\\ -2, & x=4l-1\end{cases}$