Aufgabe 14.14

Die Funktion $f(x)=\begin{cases} -1 & -\pi < x \le -\frac{\pi}{2} \\ \sin x & -\frac{\pi}{2} < x \le \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x \le \pi \end{cases}$ werde 2π–periodisch fortgesetzt.

a) Skizzieren Sie die Funktion!

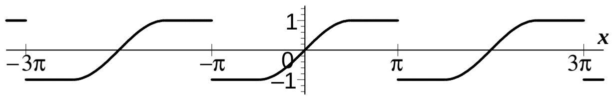
b) Entwickeln Sie die Funktion in eine Fourierreihe!

Hinweis: $\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta))$

c) Gegen welche Funktion konvergiert die Fourierreihe?

Lösung:

a) $f(x)=\begin{cases} -1 & -\pi+2k\pi < x < -\frac{\pi}{2}+2k\pi \\ \sin x & -\frac{\pi}{2}+2k\pi \le x \le \frac{\pi}{2}+2k\pi \\ 1 & \frac{\pi}{2}+2k\pi < x \le \pi+2k\pi \end{cases}, \quad k\in\mathbb{Z}$



b) $f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$

$a_k = 0$, da ungerade (Bei Berechnung wäre Integrand $f(x)\cos kx$ ungerade $\Rightarrow$ Integral 0.)

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} f(x)\sin kx\,\mathrm{d}x \text{ (da Integrand gerade)}$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \sin x \sin kx\,\mathrm{d}x + \frac{2}{\pi}\int_{\pi/2}^{\pi} \sin kx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi/2} (\cos(k-1)x - \cos(k+1)x)\,\mathrm{d}x + \frac{2}{\pi}\int_{\pi/2}^{\pi} \sin kx\,\mathrm{d}x$$

$$\left(\text{da} \quad \sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta)),\ \sin kx \sin x = \frac{1}{2}(\cos(k-1)x - \cos(k+1)x)\right)$$

$$= \frac{1}{\pi}\left[\frac{\sin(k-1)x}{k-1} - \frac{\sin(k+1)x}{k+1}\right]_0^{\pi/2} - \frac{2}{\pi}\frac{\cos kx}{k}\bigg|_{\pi/2}^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\sin(k-1)\frac{\pi}{2}}{k-1} - \frac{\sin(k+1)\frac{\pi}{2}}{k+1}\right) - \frac{2}{\pi}\frac{\cos k\pi - \cos k\frac{\pi}{2}}{k},$$ für $k=2,3,4,\ldots$, da sich für $k=1$ Division durch 0 ergeben würde

$$k=1: \quad b_1 = \frac{1}{\pi}\int\limits_0^{\pi/2}(1-\cos 2x)\,\mathrm{d}x + \frac{2}{\pi}\int\limits_{\pi/2}^{\pi}\sin x\,\mathrm{d}x$$
$$= \frac{1}{\pi}\left[x-\frac{\sin 2x}{2}\right]_0^{\pi/2} - \frac{2}{\pi}\cos x\Big|_{\pi/2}^{\pi} = \frac{1}{\pi}\cdot\frac{\pi}{2} - \frac{2}{\pi}(-1) = \frac{1}{2}+\frac{2}{\pi}$$

Für $k\geq 2$ sind zur Auswertung der Winkelfunktionen wegen der Vielfachen von $\pi/2$ vier Fälle zu unterscheiden:

$$k=4l: \quad b_k = \frac{1}{\pi}\left(\frac{\sin(4l-1)\frac{\pi}{2}}{k-1} - \frac{\sin(4l+1)\frac{\pi}{2}}{k+1}\right) - \frac{2}{\pi}\,\frac{\cos 4l\pi - \cos 4l\frac{\pi}{2}}{k}$$
$$= \frac{1}{\pi}\left(\frac{-1}{k-1} - \frac{1}{k+1}\right) - \frac{2}{\pi}\,\frac{1-1}{k} = \frac{1}{\pi}\,\frac{-k-1-k+1}{k^2-1} = -\frac{2k}{\pi(k^2-1)}$$

$$k=4l+1: \quad b_k = \frac{1}{\pi}\left(\frac{\sin 4l\frac{\pi}{2}}{k-1} - \frac{\sin(4l+2)\frac{\pi}{2}}{k+1}\right) - \frac{2}{\pi}\,\frac{\cos(4l+1)\pi - \cos(4l+1)\frac{\pi}{2}}{k} = \frac{2}{\pi k}$$

$$k=4l+2: \quad b_k = \frac{1}{\pi}\left(\frac{\sin(4l+1)\frac{\pi}{2}}{k-1} - \frac{\sin(4l+3)\frac{\pi}{2}}{k+1}\right) - \frac{2}{\pi}\,\frac{\cos(4l+2)\pi - \cos(4l+2)\frac{\pi}{2}}{k}$$
$$= \frac{1}{\pi}\left(\frac{1}{k-1} - \frac{-1}{k+1}\right) - \frac{2}{\pi}\,\frac{1-(-1)}{k} = \frac{1}{\pi}\,\frac{k+1+k-1}{k^2-1} - \frac{4}{\pi k} = \frac{2k}{\pi(k^2-1)} - \frac{4}{\pi k}$$

$$k=4l+3: \quad b_k = \frac{1}{\pi}\left(\frac{\sin(4l+2)\frac{\pi}{2}}{k-1} - \frac{\sin(4l+4)\frac{\pi}{2}}{k+1}\right) - \frac{2}{\pi}\,\frac{\cos(4l+3)\pi - \cos(4l+3)\frac{\pi}{2}}{k} = \frac{2}{\pi k}$$

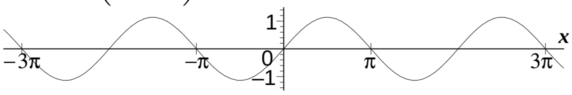
$$b_2 = \frac{4}{3\pi} - \frac{4}{2\pi} = -\frac{4}{6\pi} = -\frac{2}{3\pi}, \quad b_3 = \frac{2}{3\pi}, \quad b_4 = -\frac{8}{15\pi}, \quad b_5 = \frac{2}{5\pi}, \quad \ldots$$

Zur Probe könnte man z.B. den Koeffizienten b_2 separat berechnen. Man erhält tatsächlich

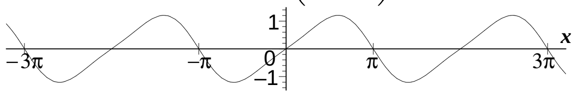
$$b_2 = \frac{1}{\pi}\int\limits_0^{\frac{\pi}{2}}(\cos x - \cos 3x)\mathrm{d}x + \frac{2}{\pi}\int\limits_{\frac{\pi}{2}}^{\pi}\sin 2x\mathrm{d}x = \frac{1}{\pi}\left[\sin x - \frac{\sin 3x}{3}\right]_0^{\frac{\pi}{2}} - \frac{1}{\pi}\cos 2x\Big|_{\frac{\pi}{2}}^{\pi}$$
$$= \frac{1}{\pi}(1-\frac{-1}{3}) - \frac{1}{\pi}(1-(-1)) = \frac{1}{\pi}\cdot\frac{4}{3} - \frac{1}{\pi}2 = -\frac{2}{3\pi}.$$

$$f(x) \sim \left(\frac{1}{2}+\frac{2}{\pi}\right)\sin x - \frac{2}{3\pi}\sin 2x + \frac{2}{3\pi}\sin 3x - \frac{8}{15\pi}\sin 4x + \frac{2}{5\pi}\sin 5x + \cdots$$
$$= \left(\frac{1}{2}+\frac{2}{\pi}\right)\sin x + \sum_{l=1}^{\infty}\left(\left(\frac{2(4l-2)}{\pi((4l-2)^2-1)} - \frac{4}{\pi(4l-2)}\right)\sin(4l-2)x\right.$$
$$\left. + \frac{2}{\pi(4l-1)}\sin(4l-1)x - \frac{2(4l)}{\pi((4l)^2-1)}\sin 4lx + \frac{2}{\pi(4l+1)}\sin(4l+1)\right)$$

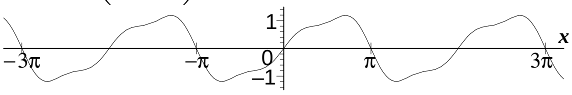
$$F_1(x)=\left(\frac{1}{2}+\frac{2}{\pi}\right)\sin x$$



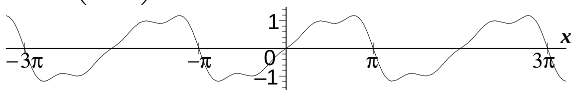
$$F_2(x)=\left(\frac{1}{2}+\frac{2}{\pi}\right)\sin x-\frac{2}{3\pi}\sin 2x$$



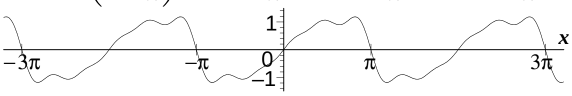
$$F_3(x)=\left(\frac{1}{2}+\frac{2}{\pi}\right)\sin x-\frac{2}{3\pi}\sin 2x+\frac{2}{3\pi}\sin 3x$$



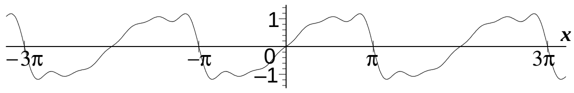
$$F_4=\left(\frac{1}{2}+\frac{2}{\pi}\right)\sin x-\frac{2}{3\pi}\sin 2x+\frac{2}{3\pi}\sin 3x-\frac{8}{15\pi}\sin 4x$$



$$F_5(x)=\left(\frac{1}{2}+\frac{2}{\pi}\right)\sin x-\frac{2}{3\pi}\sin 2x+\frac{2}{3\pi}\sin 3x-\frac{8}{15\pi}\sin 4x+\frac{2}{5\pi}\sin 5x$$



$F_6(x)$



c) Nach dem Satz von Dirichlet konvergiert die Fourierreihe gegen $f(x)$ für $x\neq(2m+1)\pi$ und gegen $\frac{1+(-1)}{2}=0$ für $x=(2m+1)\pi$.