Aufgabe 14.11

Sei $a \neq 0$. Die Funktion $f(x) = \mathrm{e}^{ax}$, $-\pi < x \leq \pi$ werde 2π–periodisch fortgesetzt.

a) Skizzieren Sie die Funktion für $a = 1/2$!
b) Entwickeln Sie die Funktion in eine Fourierreihe!
c) Gegen welche Funktion konvergiert die Fourierreihe?

Lösung:

Fourierreihen (für **periodische** Funktionen)

Periodenlänge 2π

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \,\mathrm{d}x$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \,\mathrm{d}x$$

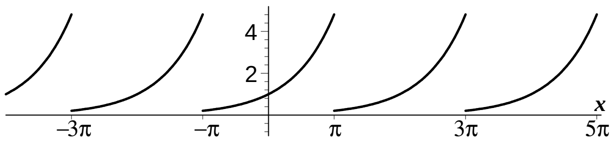
Periodenlänge T

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos k\frac{2\pi}{T}x + b_k \sin k\frac{2\pi}{T}x \right)$$

$$a_k = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \cos k\frac{2\pi}{T}x \,\mathrm{d}x$$

$$b_k = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \sin k\frac{2\pi}{T}x \,\mathrm{d}x$$

a) $\mathrm{e}^{\pi/2} \approx 4.81$
$\mathrm{e}^{-\pi/2} \approx 0.21$



Periodenlänge 2π, weder gerade noch ungerade

b) $a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \mathrm{e}^{ax} \cos kx \,\mathrm{d}x, \quad k = 0, 1, 2, \ldots$

$$\int \mathrm{e}^{ax} \cos kx \,\mathrm{d}x = \frac{1}{a}\mathrm{e}^{ax} \cos kx + \frac{k}{a} \int \mathrm{e}^{ax} \sin kx \,\mathrm{d}x$$

$$= \frac{1}{a}\mathrm{e}^{ax} \cos kx + \frac{k}{a^2}\mathrm{e}^{ax} \sin kx - \frac{k^2}{a^2} \int \mathrm{e}^{ax} \cos kx \,\mathrm{d}x$$

$$\left(1 + \frac{k^2}{a^2}\right) \int \mathrm{e}^{ax} \cos kx \,\mathrm{d}x = \frac{a\mathrm{e}^{ax} \cos kx + k\mathrm{e}^{ax} \sin kx}{a^2}$$

$$\frac{a^2 + k^2}{a^2} \int \mathrm{e}^{ax} \cos kx \,\mathrm{d}x = \frac{a\mathrm{e}^{ax} \cos kx + k\mathrm{e}^{ax} \sin kx}{a^2}$$

$$\int \mathrm{e}^{ax} \cos kx \,\mathrm{d}x = \frac{a\mathrm{e}^{ax} \cos kx + k\mathrm{e}^{ax} \sin kx}{a^2 + k^2}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \mathrm{e}^{ax} \cos kx \,\mathrm{d}x = \left.\frac{a\mathrm{e}^{ax} \cos kx + k\mathrm{e}^{ax} \sin kx}{\pi(a^2 + k^2)}\right|_{-\pi}^{\pi} = \frac{a\mathrm{e}^{a\pi} \cos k\pi - a\mathrm{e}^{-a\pi} \cos k\pi}{\pi(a^2 + k^2)}$$

$$= \frac{2a(-1)^k \sinh a\pi}{\pi(a^2 + k^2)}, \quad \text{d.h. insbesondere } a_0 = \frac{2\sinh a\pi}{a\pi} \qquad \left(\sinh t = \frac{\mathrm{e}^t - \mathrm{e}^{-t}}{2}\right)$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \mathrm{e}^{ax} \sin kx \,\mathrm{d}x, \quad k = 1, 2, \ldots$$

$$
\begin{aligned}
\int e^{ax}\sin kx\,dx &= \frac{1}{a}e^{ax}\sin kx - \frac{k}{a}\int e^{ax}\cos kx\,dx \\
&= \frac{1}{a}e^{ax}\sin kx - \frac{k}{a^2}e^{ax}\cos kx - \frac{k^2}{a^2}\int e^{ax}\sin kx\,dx
\end{aligned}
$$

$$
\left(1+\frac{k^2}{a^2}\right)\int e^{ax}\sin kx\,dx = \frac{ae^{ax}\sin kx - ke^{ax}\cos kx}{a^2}
$$

$$
\frac{a^2+k^2}{a^2}\int e^{ax}\sin kx\,dx = \frac{ae^{ax}\sin kx - ke^{ax}\cos kx}{a^2}
$$

$$
\int e^{ax}\sin kx\,dx = \frac{ae^{ax}\sin kx - ke^{ax}\cos kx}{a^2+k^2}
$$

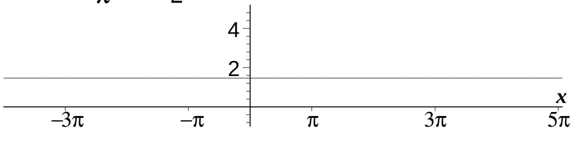
$$
\begin{aligned}
b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\sin kx\,dx &= \frac{ae^{ax}\sin kx - ke^{ax}\cos kx}{\pi(a^2+k^2)}\bigg|_{-\pi}^{\pi} = \frac{-ke^{a\pi}\cos k\pi + ke^{-a\pi}\cos k\pi}{\pi(a^2+k^2)} \\
&= -\frac{2k(-1)^k\sinh a\pi}{\pi(a^2+k^2)}
\end{aligned}
$$

Fourierentwicklung also

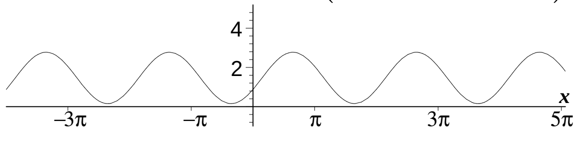
$$
\begin{aligned}
f(x) &\sim \frac{\sinh a\pi}{a\pi} + \sum_{k=1}^{\infty}\left(\frac{2a(-1)^k\sinh a\pi}{\pi(a^2+k^2)}\cos kx - \frac{2k(-1)^k\sinh a\pi}{\pi(a^2+k^2)}\sin kx\right) \\
&\sim \frac{2}{\pi}\sinh a\pi\left(\frac{1}{2a} + \sum_{k=1}^{\infty}\frac{(-1)^k}{a^2+k^2}(a\cos kx - k\sin kx)\right) \\
&\sim \frac{2}{\pi}\sinh a\pi\left(\frac{1}{2a} + \left[-\frac{1}{a^2+1}(a\cos x - \sin x) + \frac{1}{a^2+4}(a\cos 2x - 2\sin 2x) \mp \dots\right]\right)
\end{aligned}
$$

$$
F_n(x) = \frac{2}{\pi}\sinh\frac{\pi}{2}\left(1 + \sum_{k=1}^{n}\frac{(-1)^k}{k^2+1/4}\left(\frac{1}{2}\cos kx - k\sin kx\right)\right)
$$

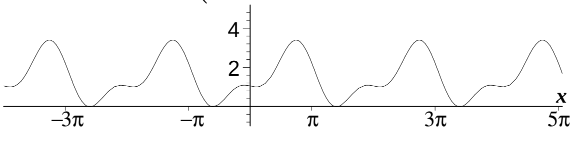
$F_0(x) = \frac{2}{\pi}\sinh\frac{\pi}{2}$



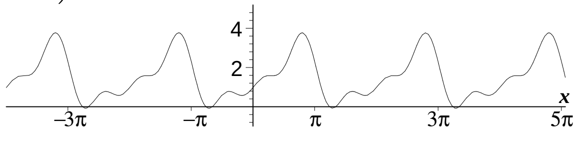
$F_1(x) = \frac{2}{\pi}\sinh\frac{\pi}{2}\left(1 - \frac{(\cos x)/2 - \sin x}{1.25}\right)$



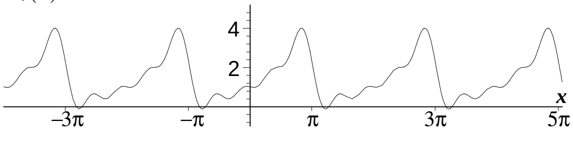
$F_2(x) = \frac{2}{\pi}\sinh\frac{\pi}{2}\left(1 - \frac{(\cos x)/2 - \sin x}{1.25} + \frac{(\cos 2x)/2 - 2\sin 2x}{4.25}\right)$



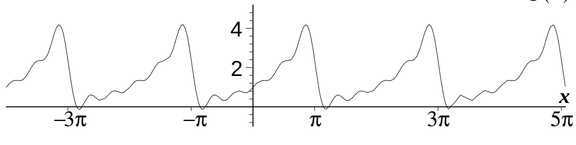
$F_3(x)$



$F_4(x)$



$F_5(x)$



c) Satz von Dirichlet: In Stetigkeitspunkten Konvergenz gegen $f(x)$,

an Sprungstellen gegen $\dfrac{f(x+0)+f(x-0)}{2}$.

Also: $$F_n(x) \xrightarrow[n\to\infty]{} \begin{cases} f(x), & x \neq (2l+1)\pi \\ \dfrac{e^{a\pi}+e^{-a\pi}}{2} = \cosh a\pi, & x = (2l+1)\pi \end{cases}.$$