Aufgabe 14.10

Sie Funktion $f(t) = \mathrm{e}^t,\ -\pi < t \le \pi$ werde 2π–periodisch fortgesetzt.

a) Skizzieren Sie die Funktion!
b) Entwickeln Sie die Funktion in eine Fourierreihe!
c) Gegen welche Funktion konvergiert die Fourierreihe?

Lösung:

Fourierreihen (für **periodische** Funktionen)

Periodenlänge 2π

$$f(t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt)$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \,\mathrm{d}t$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \,\mathrm{d}t$$

Periodenlänge T

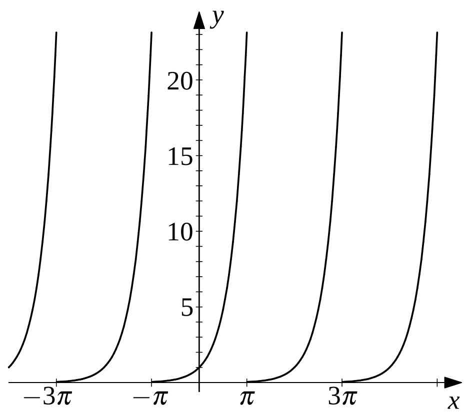
$$f(t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos k\frac{2\pi}{T}t + b_k \sin k\frac{2\pi}{T}t \right)$$

$$a_k = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos k\frac{2\pi}{T}t \,\mathrm{d}t$$

$$b_k = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin k\frac{2\pi}{T}t \,\mathrm{d}t$$

a) $\mathrm{e}^{\pi} \approx 23.14$
$\mathrm{e}^{-\pi} \approx 0.04$

Periodenlänge 2π, weder gerade noch ungerade



b) $a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \mathrm{e}^t \cos kt \,\mathrm{d}t, \quad k = 0, 1, 2, \ldots$

$$\int \mathrm{e}^t \cos kt \,\mathrm{d}t = \mathrm{e}^t \cos kt + k \int \mathrm{e}^t \sin kt \,\mathrm{d}t$$

$$= \mathrm{e}^t \cos kt + k\mathrm{e}^t \sin kt - k^2 \int \mathrm{e}^t \cos kt \,\mathrm{d}t$$

$$(1+k^2) \int \mathrm{e}^t \cos kt \,\mathrm{d}t = \mathrm{e}^t \cos kt + k\mathrm{e}^t \sin kt$$

$$\int \mathrm{e}^t \cos kt \,\mathrm{d}t = \frac{\mathrm{e}^t \cos kt + k\mathrm{e}^t \sin kt}{1+k^2}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \mathrm{e}^t \cos kt \,\mathrm{d}t = \left. \frac{\mathrm{e}^t \cos kt + k\mathrm{e}^t \sin kt}{\pi(1+k^2)} \right|_{-\pi}^{\pi} = \frac{\mathrm{e}^{\pi} \cos k\pi - \mathrm{e}^{-\pi} \cos k\pi}{\pi(1+k^2)}$$

$$= \frac{2(-1)^k \sinh \pi}{\pi(1+k^2)}, \quad \text{d.h. insbesondere } a_0 = \frac{2 \sinh \pi}{\pi} \qquad \left(\sinh t = \frac{\mathrm{e}^t - \mathrm{e}^{-t}}{2} \right)$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{e}^t \sin kt \, \mathrm{d}t, \quad k = 1, 2, \ldots$$

$$\int \mathrm{e}^t \sin kt \, \mathrm{d}t = \mathrm{e}^t \sin kt - k\int \mathrm{e}^t \cos kt \, \mathrm{d}t$$

$$= \mathrm{e}^t \sin kt - k\,\mathrm{e}^t \cos kt - k^2 \int \mathrm{e}^t \sin kt \, \mathrm{d}t$$

$$\left(1+k^2\right)\int \mathrm{e}^t \sin kt \, \mathrm{d}t = \mathrm{e}^t \sin kt - k\mathrm{e}^t \cos kt$$

$$\int \mathrm{e}^t \sin kt \, \mathrm{d}t = \frac{\mathrm{e}^t \sin kt - k\mathrm{e}^t \cos kt}{1+k^2}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{e}^t \sin kt \, \mathrm{d}t = \left.\frac{\mathrm{e}^t \sin kt - k\mathrm{e}^t \cos kt}{\pi(1+k^2)}\right|_{-\pi}^{\pi} = \frac{-k\mathrm{e}^{\pi} \cos k\pi + k\mathrm{e}^{-\pi} \cos k\pi}{\pi(1+k^2)}$$

$$= -\frac{2k(-1)^k \sinh \pi}{\pi(1+k^2)}$$

Fourierentwicklung also

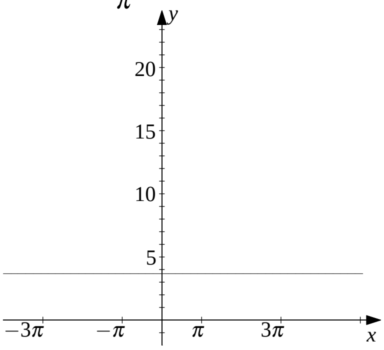
$$f(t) \sim \frac{\sinh \pi}{\pi} + \sum_{k=1}^{\infty} \left(\frac{2(-1)^k \sinh a\pi}{\pi(1+k^2)} \cos kt - \frac{2k(-1)^k \sinh \pi}{\pi(1+k^2)} \sin kt \right)$$

$$\sim \frac{2}{\pi} \sinh \pi \left(\frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k}{1+k^2} (\cos kt - k \sin kt) \right)$$

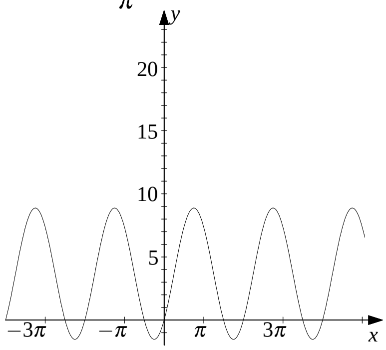
$$\sim \frac{2}{\pi} \sinh \pi \left(\frac{1}{2} + \left[-\frac{1}{2}(\cos t - \sin t) + \frac{1}{5}(\cos 2t - 2\sin 2t) \mp \ldots \right] \right)$$

$$F_n(t) = \frac{2}{\pi} \sinh \pi \left(\frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k}{1+k^2} (\cos kt - k \sin kt) \right)$$

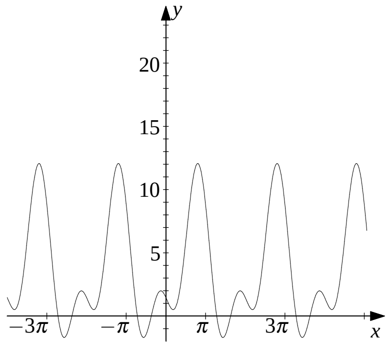
$F_0(t) = \dfrac{\sinh\pi}{\pi}$



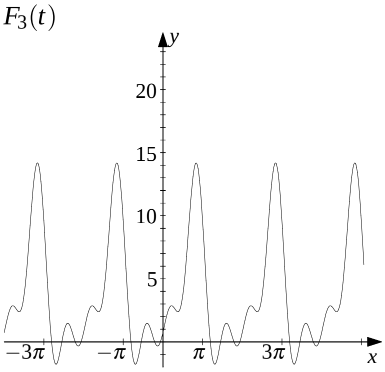
$F_1(t) = \dfrac{\sinh\pi}{\pi}\,(1-\cos t+\sin t)$



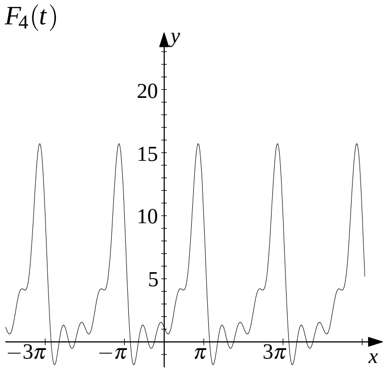
$F_2(t)$



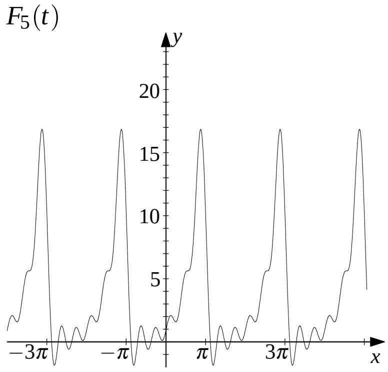
$F_3(t)$



$F_4(t)$



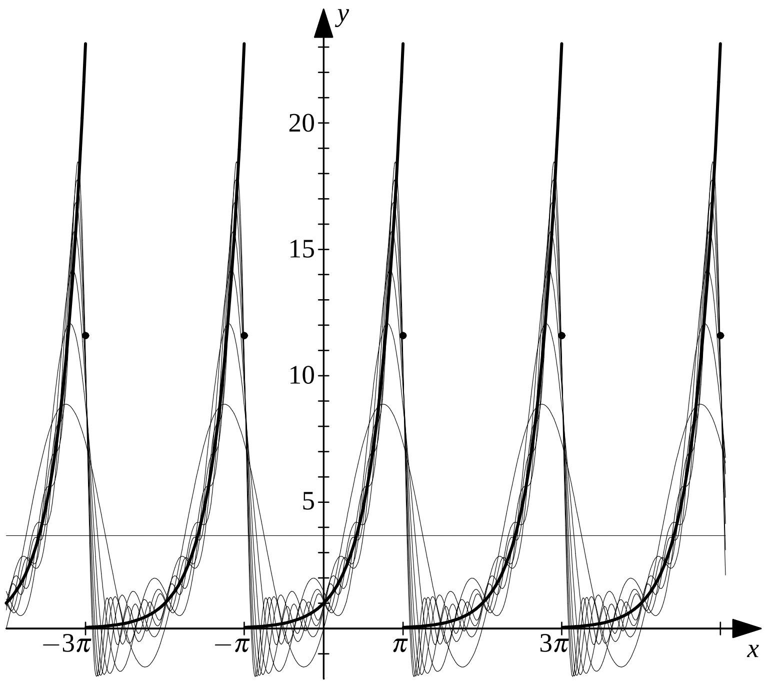
$F_5(t)$



c) Satz von Dirichlet: In Stetigkeitspunkten Konvergenz gegen $f(t)$,

an Sprungstellen gegen $\dfrac{f(t+0)+f(t-0)}{2}$.

Also: $$F_n(t) \xrightarrow[n\to\infty]{} \begin{cases} f(t), & t \neq (2l+1)\pi \\ \dfrac{e^{\pi}+e^{-\pi}}{2} = \cosh\pi, & t = (2l+1)\pi \end{cases}.$$



$f(t)$ und $F_0(t)$ bis $F_7(t)$