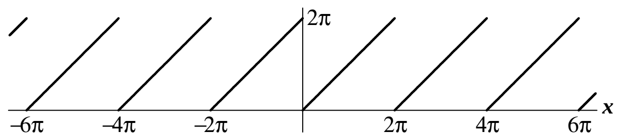
Aufgabe 14.9

Die Funktion $f(x) = x,\ 0 \le x < 2\pi$ werde periodisch fortgesetzt.

a) Skizzieren Sie die Funktion!
b) Entwickeln Sie die Funktion in eine Fourierreihe!
c) Gegen welche Funktion konvergiert die Fourierreihe?

Lösung:

a)



b) Periodenlänge 2π, $\quad f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$,

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\cos kx\,\mathrm{d}x,\ \ b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin kx\,\mathrm{d}x$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,\mathrm{d}x = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos kx\,\mathrm{d}x = \frac{1}{\pi}\int_{0}^{2\pi} x\cos kx\,\mathrm{d}x \ \ \text{(wegen Periodizität)}$$

$$\int x\cos kx\,\mathrm{d}x = x\frac{\sin kx}{k} - \int \frac{\sin kx}{k}\mathrm{d}x = \frac{x\sin kx}{k} + \frac{\cos kx}{k^2}, \ \ \text{falls } k \ne 0$$

$$k=0: \ \ a_0 = \frac{1}{\pi}\int_0^{2\pi} x\,\mathrm{d}x = \frac{1}{\pi}\frac{x^2}{2}\bigg|_0^{2\pi} = 2\pi$$

$$k\ne 0: \ \ a_k = \frac{1}{\pi}\int_0^{2\pi} x\cos kx\,\mathrm{d}x = \frac{1}{\pi}\left(\frac{x\sin kx}{k} + \frac{\cos kx}{k^2}\right)\bigg|_0^{2\pi} = \frac{1}{\pi}\left(0 + \frac{1}{k^2} - 0 - \frac{1}{k^2}\right) = 0$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,\mathrm{d}x = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin kx\,\mathrm{d}x = \frac{1}{\pi}\int_{0}^{2\pi} x\sin kx\,\mathrm{d}x \ \ \text{(wegen Periodizität)}$$

$$\int x\sin kx\,\mathrm{d}x = x\left(-\frac{\cos kx}{k}\right) + \int \frac{\cos kx}{k}\mathrm{d}x = -\frac{x\cos kx}{k} + \frac{\sin kx}{k^2}$$

$k=0:$ für Sinus–Koffizient nicht möglich.

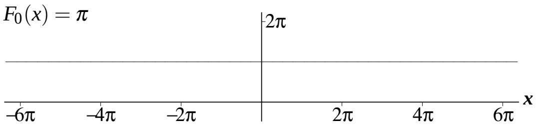
$$k\ne 0: \ b_k = \frac{1}{\pi}\int_0^{2\pi} x\sin kx\,\mathrm{d}x = \frac{1}{\pi}\left(-\frac{x\cos kx}{k} + \frac{\sin kx}{k^2}\right)\bigg|_0^{2\pi} = \frac{1}{\pi}\left(-\frac{2\pi}{k} + 0 + 0 - 0\right) = -\frac{2}{k}$$

Fourierentwicklung also

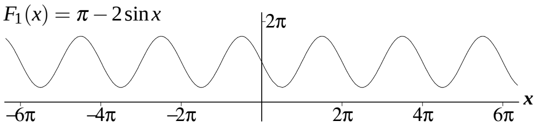
$$f(x) \sim \pi - \sum_{k=1}^{\infty}\frac{2}{k}\sin kx = \pi - 2\sum_{k=1}^{\infty}\frac{\sin kx}{k} = \pi - 2\left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right)$$

$$F_n(x) = \pi - 2\sum_{k=1}^{n}\frac{\sin kx}{k}$$

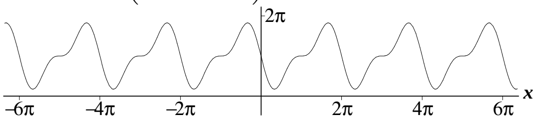
$F_0(x) = \pi$



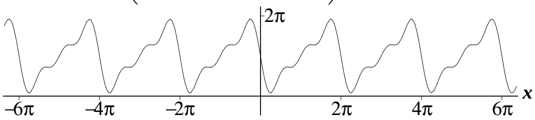
$F_1(x) = \pi - 2\sin x$



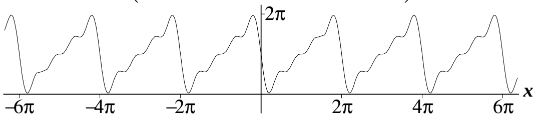
$F_2(x) = \pi - 2\left(\sin x + \frac{\sin 2x}{2}\right)$



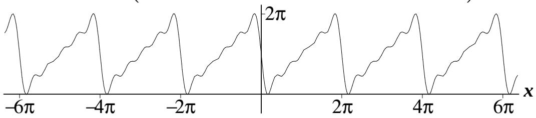
$F_3(x) = \pi - 2\left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3}\right)$



$F_4(x) = \pi - 2\left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4}\right)$



$F_5(x) = \pi - 2\left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4} + \frac{\sin 5x}{5}\right)$



c) Satz von Dirichlet: In Stetigkeitspunkten Konvergenz gegen $f(x)$,

an Sprungstellen gegen $\dfrac{f(x+0) + f(x-0)}{2}$.

Also: $$F_n(x) \xrightarrow[n \to \infty]{} \begin{cases} f(x), & x \neq 2l\pi \\ \dfrac{2\pi + 0}{2} = \pi, & x = 2l\pi \end{cases}.$$