Aufgabe 13.94

Berechnen Sie folgende uneigentliche Integrale:

a) $\int\limits_1^2 \frac{dx}{\sqrt[3]{x-1}}$, b) $\int\limits_2^\infty \frac{dx}{\sqrt[3]{x-1}}$, c) $\int\limits_0^{\pi/2} \tan x \, dx$, d) $\int\limits_{-\infty}^{2\sqrt{3}} \frac{dx}{x^2+4}$!

Lösung:

a) $\int\limits_1^2 \frac{dx}{\sqrt[3]{x-1}} = \int\limits_1^2 (x-1)^{-\frac{1}{3}} \, dx = \frac{3}{2}(x-1)^{\frac{2}{3}}\Big|_1^2 = \frac{3}{2} - 0 = \frac{3}{2}$

b) $\int\limits_2^\infty \frac{dx}{(x-1)^3} = \int\limits_2^\infty (x-1)^{-\frac{1}{3}} \, dx = \frac{3}{2}(x-1)^{\frac{2}{3}}\Big|_2^\infty = \infty - \frac{3}{2} = \infty$

c) $\int\limits_0^{\pi/2} \tan x \, dx = \int\limits_{x=0}^{x=\frac{\pi}{2}} \frac{\sin x \, dx}{\cos x} = \int\limits_{t=1}^{t=0} \frac{-dt}{t} = -\ln t|_1^0 = -(-\infty) - 0 = \infty$

$\left(\text{Substitution } t = \cos x, \ \frac{dt}{dx} = -\sin x, \ \sin x \, dx = -dt\right)$

d) $\int \frac{1}{x^2+4} \, dx = \frac{1}{4}\int \frac{dx}{\frac{x^2}{4}+1} = \frac{1}{4}\int \frac{dx}{\left(\frac{x}{2}\right)^2+1} = \frac{1}{4}\int \frac{2\,dt}{t^2+1} = \frac{1}{2}\int \frac{dt}{t^2+1} = \frac{1}{2}\arctan t + C = \frac{1}{2}\arctan\frac{x}{2} + C$

$\left(\text{Substitution: } t = \frac{x}{2}, \ \frac{dt}{dx} = \frac{1}{2}, \ dx = 2\,dt\right)$

$\int\limits_{-\infty}^{2\sqrt{3}} \frac{dx}{x^2+4} = \frac{1}{2}\arctan\frac{x}{2}\Big|_{-\infty}^{2\sqrt{3}} = \frac{1}{2}\arctan\sqrt{3} - \frac{1}{2}\arctan(-\infty) = \frac{1}{2}\frac{\pi}{3} - \frac{1}{2}\left(-\frac{\pi}{2}\right) = \frac{2\pi+3\pi}{12} = \frac{5\pi}{12}$