Aufgabe 13.24

Berechnen Sie mithilfe geeigneter Substitutionen

a) $\displaystyle\int\frac{dx}{e^{3x}+5}$, b) $\displaystyle\int\frac{dx}{\sin x}$, c) $\displaystyle\int\frac{dx}{8-4\sin x+7\cos x}$, d) $\displaystyle\int\frac{\sqrt{1+x}}{x}\,dx$,

e) $\displaystyle\int\sqrt{\frac{1-x}{1+x}}\frac{dx}{(1-x)^2}$, f) $\displaystyle\int\sin 4x\sin 6x\,dx$!

Lösung:

a) $t=e^{3x}$, $\displaystyle\int\frac{dx}{e^{3x}+5}=\int\frac{\frac{1}{3t}\,dt}{t+5}=\frac{1}{15}\int\left(\frac{1}{t}-\frac{1}{t+5}\right)dt=\frac{1}{15}\ln C\left(\frac{e^{3x}}{e^{3x}+5}\right)$

b) $t=\tan\frac{x}{2}$, $\frac{x}{2}=\arctan t$, $x=2\arctan t$, $\frac{dx}{dt}=\frac{2}{1+t^2}$, $dx=\frac{2\,dt}{1+t^2}$

$$\sin x=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}=\frac{\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}}}{\frac{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}=\frac{2t}{1+t^2},\quad \cos x=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}$$

$$\int\frac{dx}{\sin x}=\int\frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}}=\int\frac{dt}{t}=\ln|t|+C=\ln\left|\tan\frac{x}{2}\right|+C$$

c) $t=\tan\frac{x}{2}$, $\displaystyle\int\frac{dx}{8-4\sin x+7\cos x}=\int\frac{2dt}{8(1+t^2)-4\cdot 2t+7(1-t^2)}=\int\frac{2dt}{t^2-8t+15}$

$$=\int\left(\frac{dt}{t-5}-\frac{dt}{t-3}\right)=\ln C\left|\frac{\tan\frac{x}{2}-5}{\tan\frac{x}{2}-3}\right|$$

d) $t=\sqrt{1+x}$, $x=t^2-1$, $\displaystyle\int\frac{\sqrt{1+x}}{x}dx=2\int\frac{t^2}{t^2-1}dt=2\int\left(1+\frac{1}{t^2-1}\right)dt$

$$=2\sqrt{1+x}+\ln\left|\frac{1-\sqrt{1+x}}{1+\sqrt{1+x}}\right|+C$$

e) $t=\sqrt{\frac{1-x}{1+x}}$, $dx=-\frac{4t}{(1+t^2)^2}dt$, $\displaystyle\int\sqrt{\frac{1-x}{1+x}}\frac{dx}{(1-x)^2}=-\int\frac{dt}{t^2}=\sqrt{\frac{1+x}{1-x}}+C$

f) $\displaystyle\int\sin 4x\sin 6x\,dx=\frac{1}{2}\int(\cos 2x-\cos 10x)\,dx=\frac{1}{4}\sin 2x-\frac{1}{20}\sin 10x+C$