Aufgabe 9.46

Berechnen Sie S_{20} und falls existent $\lim\limits_{n\to\infty} S_n$ für

a) $S_n = \sum\limits_{k=3}^{n} 2^k 3^{-k}$, b) $S_n = \sum\limits_{k=3}^{n} 2^{-k} 3^k$, c) $S_n = \sum\limits_{k=3}^{n} \left(2-2^{-k}\right)$!

Lösung:

a) $$S_n = \sum_{k=3}^{n} 2^k 3^{-k} = \sum_{k=3}^{n} \left(\frac{2}{3}\right)^k = \sum_{k=0}^{n-3} \left(\frac{2}{3}\right)^{k+3} = \frac{2^3}{3^3} \sum_{k=0}^{n-3} \left(\frac{2}{3}\right)^k = \frac{8}{27} \frac{1-\left(\frac{2}{3}\right)^{(n-3)+1}}{1-\frac{2}{3}}$$
$$= \frac{8}{9}\left(1-\left(\frac{2}{3}\right)^{n-2}\right), \quad S_{20} = \frac{8}{9}\left(1-\left(\frac{2}{3}\right)^{18}\right) \approx 0.88828743$$

Wegen $\left|\frac{2}{3}\right| < 1$ gilt $\lim\limits_{n\to\infty} S_n = \sum\limits_{k=3}^{\infty} 2^k 3^{-k} = \frac{8}{9} \approx 0.88888888$.

b) $$S_n = \sum_{k=3}^{n} 2^{-k} 3^k = \sum_{k=3}^{n} \left(\frac{3}{2}\right)^k = \sum_{k=0}^{n-3} \left(\frac{3}{2}\right)^{k+3} = \frac{3^3}{2^3} \sum_{k=0}^{n-3} \left(\frac{3}{2}\right)^k = \frac{27}{8} \frac{1-\left(\frac{3}{2}\right)^{(n-3)+1}}{1-\frac{3}{2}}$$
$$= \frac{27}{4}\left(\left(\frac{3}{2}\right)^{n-2} - 1\right), \quad S_{20} = \frac{27}{4}\left(\left(\frac{3}{2}\right)^{18} - 1\right) \approx 9969.0202$$

Wegen $\frac{3}{2} > 1$ gilt $\lim\limits_{n\to\infty} S_n = \sum\limits_{k=3}^{\infty} 2^{-k} 3^k = \infty$, die Reihe divergiert bestimmt.

c) $$S_n = \sum_{k=3}^{n} \left(2-2^{-k}\right) = (n-2)\cdot 2 - \sum_{k=3}^{n} \left(\frac{1}{2}\right)^k = 2n-4-\sum_{k=0}^{n-3} \left(\frac{1}{2}\right)^{k+3}$$
$$= 2n-4-\frac{1}{2^3}\sum_{k=0}^{n-3} \left(\frac{1}{2}\right)^k = 2n-4-\frac{1}{2^3}\frac{1-\frac{1}{2^{n-2}}}{1-\frac{1}{2}} = 2n-4-\frac{1}{4}\left(1-\frac{4}{2^n}\right)$$
$$= 2n-4-\frac{1}{4}+\frac{1}{2^n} = 2n-\frac{17}{4}+\frac{1}{2^n}, \quad S_{20} = 40-4.25+\frac{1}{2^{20}} \approx 35.75000095$$

Wegen $S_n > 2n-\frac{17}{4} \longrightarrow \infty$ für $n\to\infty$ gilt auch $\lim\limits_{n\to\infty} S_n = \infty$, die Reihe divergiert bestimmt.